HIGHER ALGEBRA

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HIGHER ALGEBRA

BY

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PREFACE

This text is prepared to meet the needs of the student who will continue his mathematics as far as the calculus, and is written in the spirit of applied mathematics. This does not imply that algebra for the engineer is a different subject from algebra for the college man or for the secondary student who is prepared to take such a course. In fact, the topics which the engineer must emphasize, such as numerical computations, checks, graphical methods, use of tables, and the solution of specific problems, are among the most vital features of the subject for any student. But important as these topics are, they do not comprise the substance of algebra, which enables it to serve as part of the foundation for future work. Rather they furnish an atmosphere in which that foundation may be well and intelligently laid.

The concise review contained in the first chapter covers the topics which have direct bearing on the work which follows. No attempt is made to repeat all of the definitions of elementary algebra. It is assumed that the student retains a certain residue from his earlier study of the subject.

The quadratic equation is treated with unusual care and thoroughness. This is done not only for the purpose of review, but because a mastery of the theory of this equation is absolutely necessary for effective work in analytical geometry and calculus. Furthermore, a student who is well grounded in this particular is in a position to appreciate the methods and results of the theory of the general equation with a minimum of effort.

The theory of equations forms the keystone of most courses in higher algebra. The chapter on this subject is developed gradually, and yet with pointed directness, in the hope that the processes which students often perform in a perfunctory manner will take on additional life and interest.

Throughout the text the attempt is made to anticipate the difficulties of the student, and by the use of illustrative material to make the book readable, incidentally reducing the labor of exposition on

7737

the part of the instructor. In this connection §§ 18, 19, 69, and 89 may be consulted as furnishing instances of the method of procedure.

The exercises are for the most part new, and serve not only to illustrate the text but to test and develop the power of the student at every turn.

The author is under obligation to Professors G. B. Pegram and C. H. Burnside for exercises from their special fields of science. Especial acknowledgment should be made to my colleague, Dr. H. W. Reddick, who has prepared a large part of the collection of exercises, and whose criticisms, both destructive and constructive, have been invaluable throughout the preparation of the book.

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CONTENTS

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CHAPTER I

INTRODUCTORY REVIEW

1. Factoring. The process of **factoring** consists in finding two or more expressions whose product is equal to a given expression.

In the type forms which are considered in this section it is assumed that all of the coefficients are integers. Unless the contrary is stated, only factors having integral coefficients are required. A **prime** expression has no factors with integral coefficients, except itself and 1.

Later in this text it will be necessary to find factors whose coefficients are not integral, but irrational, or even complex numbers. In every case of this kind the nature of the problem in hand will indicate the type of factors desired.

The following suggestions will prove helpful in factoring:

I. *First look for a monomial factor common to every term of the given expression. If one exists, separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then determine, from the form of the polynomial factor, with which of the following types it should be classed, and use the method of factoring applicable to that type.*

III. *Proceed again as in II with each polynomial factor obtained, until the original expression has been separated into its prime factors.*

TYPE FORMS

1. $ax + ay + bx + by = (a + b)(x + y)$.

2. $a^2 + 2\,ab + b^2 = (a + b)(a + b)$.

3. $x^2 + bx + c = (x + p)(x + q)$,

where p and q are two numbers whose sum is b and whose product is c.

4. $ax^2 + bx + c.$

To factor expressions of this type, find two numbers whose algebraic sum is b and whose product is $a \cdot c$. Replace bx by two terms in x whose respective coefficients are the numbers just found, and factor by grouping terms.

$$\text{Thus} \quad 6x^2 - 13x - 5 = 6x^2 - 15x + 2x - 5$$
$$= 3x(2x-5) + (2x-5) = (3x+1)(2x-5).$$

5. $a^2 - b^2 = (a+b)(a-b).$

6. $a^4 + ka^2b^2 + b^4.$

This type can sometimes be reduced to type 5 by adding and subtracting a multiple of a^2b^2.

7. $$\begin{cases} a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}), \\ a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}), \end{cases}$$

when n is odd. If n is even, $a^n - b^n$ is the difference of two squares (type 5). In all other cases, if n is a multiple of 3, apply one of the special types

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

8. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = (a+b+c)^2.$

EXERCISES

Factor:

1. $x^2 - kx + 2lx - 2kl.$
2. $a^2 - y^2 + 2y - 1.$
3. $r^2 - 10r - 24.$
4. $338x^2 - 52xz + 2z^2.$
5. $(x-1)(2x-3) - 6.$
6. $x - x^7.$
7. $32 - 2z^4.$
8. $a^7 + b^7.$
9. $a^9 - b^9.$

   Solution: $a^9 - b^9 = (a^3)^3 - (b^3)^3$
   $= (a^3 - b^3)(a^6 + a^3b^3 + b^6)$
   $= (a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)$
10. $a^{12} + b^{12}.$
11. $x^{10} - y^{15}.$
12. $a^2b^3 - x^2y^3 + a^2y^3 - b^3x^2.$
13. $b^2 - x^2 + 4a(a-b).$
14. $x^2 - 4a^2 + 9y^2 - b^2 - 6xy - 4ab.$
15. $2(a^3 - 1) + 7(a^2 - 1).$
16. $abx^2 + aby^2 - (a^2 + b^2)xy.$
17. $a^2 + 9b^2 + 25c^2 - 6ab - 10ac + 30bc.$

18. $(s^2-4)^2-(s+2)^2$.

19. $x^3-7x^2+14x-8$.

20. $x^8-7x^4y^4+y^8$.

21. x^3-4x^2+8.

22. $51x^2+113xy-14y^2$.

23. $8x^4y^2-65x^2y^2z+8y^2z^2$.

24. $2x^{12}-128$.

25. $(x-1)(x-2)(x-3)(x-4)-24$.

2. Simplification of fractions. To simplify an expression containing fractions, it must be reduced to a simple fraction in which the numerator and the denominator have no common factor. In reducing fractions to their simplest forms, one merely performs the indicated operations as directly as possible.

As a general rule, fractions with related denominators should be combined and the result reduced to its lowest terms before the whole expression is written as a single fraction. It is very desirable to be on the alert for opportunities to cancel factors from numerator and denominator of a given fraction. For this purpose the processes of factoring should be at ready command.

EXERCISES

Simplify:

1. $\dfrac{1-\frac{1}{2}+\frac{1}{4}}{1+\frac{1}{4}+\frac{1}{16}}$.

2. $\dfrac{\frac{1}{27}-(\frac{2}{3})^2}{\frac{11}{3}}+2\frac{1}{3}$.

3. $\dfrac{3\,(3.4-1.6)}{.027}\div\dfrac{(1.3)^2-(1.2)^2}{.1}$.

4. $1-\dfrac{2}{3-\frac{4}{5}}$.

5. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}$.

6. $1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+1}}}$.

7. $\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{a-b}{2\,(a+b)}+\dfrac{a}{b-a}+2$.

8. $\dfrac{a-b}{2\,(a+b)}-\dfrac{a^2+b^2}{b^2-a^2}-\dfrac{a}{a-b}+\dfrac{2\,b}{a+b}$.

9. $3-\cfrac{2}{5+\cfrac{4}{7+\cfrac{6}{x}}}$.

10. $\cfrac{1}{x^2-\cfrac{x^3-1}{x+\cfrac{1}{x+1}}}$.

11. $\dfrac{2}{x-1}-\dfrac{2}{x}-\dfrac{1}{x^2}-\dfrac{1}{x^3}$.

12. $\dfrac{2}{5x^2-10x+5}+\dfrac{11}{25x-25}-\dfrac{11x-4}{25x^2+100}$.

13. $\dfrac{m-n-\dfrac{2n(m-n)}{m+n}}{\dfrac{m^2+n^2}{mn+n^2}-1}$.

14. $\dfrac{a-b-\dfrac{a^2b-ab^2}{(a+b)^2}}{\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}}$.

3. Roots and radicals. In applied mathematics the square roots or the cube roots of numbers are usually found from tables of square roots or cube roots, like those on page 215, or by use of the slide rule. In problems where great accuracy is desired, more extensive tables may be used, the roots may be found by the use of logarithms, or, as a last resort, the root may actually be extracted by the rule found in the more elementary books on algebra. This rule finds its chief usefulness, however, in the extraction of roots of algebraic expressions.

The relation between the radical and the exponential notation is expressed by the formula

$$\sqrt[b]{x^a}=(\sqrt[b]{x})^a=x^{\frac{a}{b}},$$

where a and b are assumed to be integers, and b is not zero.

The only exception to this assumption found in the present text appears in the chapter on logarithms, where irrational exponents are used.

In a fractional exponent the numerator indicates the power to which the number is to be raised, while the denominator gives the index of the root which is to be extracted.

4. Fundamental laws of exponents. The laws of exponents may be stated as follows:

I. Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

II. Law of Division, $x^a \div x^b = x^{a-b}$.

III. Law of Involution, or raising to a power,

$$(x^b)^a = (x^a)^b = x^{ab}.$$

In these three laws the letters a and b may have any real value. But the only occasion which we shall have in this book to consider any exponents more complicated than rational fractions will arise in the chapter on logarithms.

An important special case arises under Law II when $a = b$. We then have

$$1 = x^a \div x^a = x^{a-a} = x^0,$$

which defines the meaning of a zero exponent. Expressed in words, this means that any number raised to the zero power equals unity.

When $a = 0$, Law II defines the meaning of a negative exponent; namely, $\frac{1}{x^b} = x^{-b}$ and gives rise to the following rule for getting rid of negative exponents in any algebraic expression.

*Any **factor** of the numerator of a fraction may be taken from the numerator and written as a **factor** of the denominator, and vice versa, if the sign of the exponent of the factor be changed.*

Thus $\quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}; \quad \frac{2x^{-2}}{\sqrt[4]{16^{-3}}} = \frac{2x^{-2}}{16^{-\frac{3}{4}}} = \frac{2 \cdot 16^{\frac{3}{4}}}{x^2} = \frac{2 \cdot 8}{x^2} = \frac{16}{x^2}.$

It is necessary to observe carefully whether the expression affected by a negative exponent is a *factor* of the numerator or denominator before transferring it.

Thus $\quad \frac{a^{-1} + b}{c} = \frac{\frac{1}{a} + b}{c} = \frac{1 + ab}{ac}$, while $\frac{a^{-1}b}{c} = \frac{b}{ac}.$

In this text the symbol $\sqrt{a}$ will be used with the single meaning $+\sqrt{a}$, not $-\sqrt{a}$. If both values of the square root of a number are intended, both the plus and the minus sign will be written.

5. Rationalization. If the product of two irrational expressions is rational, each is called a **rationalizing factor** of the other.

Thus $\sqrt{2}$ multiplied by $\sqrt{2}$ gives the rational product 2. Hence they are rationalizing factors of each other. The numbers $-\sqrt{2}$ and $\sqrt{8}$ are also rationalizing factors of $\sqrt{2}$. Similarly, $a + \sqrt{b}$ is a rationalizing factor of $a - \sqrt{b}$.

The process of rationalizing the denominator of a fraction consists in multiplying both terms of the fraction by a rationalizing factor of the denominator, so that in the simplified result the new denominator will be rational.

For purposes of computation it is convenient to rationalize the denominator of a fraction, since we are then able to compute the approximate value of the fraction much more rapidly.

The numerators of fractions are also often rationalized. In this process we multiply both terms of the fraction by a rationalizing factor of the numerator.

EXERCISES

1. Is $\sqrt{2}+\sqrt{3}=\sqrt{5}$? Is $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$?

2. Is $\sqrt{a^2+b^2}=a+b$? Is $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$?

3. Is $\sqrt{x^2+a}=x\sqrt{1+a}$? Is $\sqrt{x^2+ax^2}=x\sqrt{1+a}$?

4. Is $x^{-2}+y^{-2}=\dfrac{1}{x^2+y^2}$? Is $x^{-2}\cdot y^{-2}=\dfrac{1}{x^2y^2}$?

Simplify the following:

5. $\sqrt{50}-\sqrt{32}+\sqrt{98}$.

6. $\sqrt[3]{16}+\sqrt{8}-\sqrt[6]{4}$.

7. $4\sqrt{\frac{15}{2}}-\sqrt{\frac{3}{10}}+\sqrt{20}$.

8. $\dfrac{-2+\sqrt{12}}{2}+\dfrac{3-\sqrt{162}}{6}$.

9. $\dfrac{a-x+\sqrt{(a-x)^2+4ax}}{2}$.

10. $\sqrt{\dfrac{a}{c}}-\sqrt{\dfrac{c}{a}}+\sqrt{\dfrac{a^2+c^2}{ac}+2}-\sqrt{\dfrac{a^2+c^2}{ac}-2}$.

11. $\sqrt{3}\cdot\sqrt[3]{2}+\sqrt[3]{12}\cdot\sqrt{\frac{1}{8}}\cdot\sqrt[6]{6}$.

12. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}\div\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

13. Show that $\frac{1}{6}\left(5\pm\sqrt{109}\right)$ satisfies the equation $3x^2-5x-7=0$, if it be substituted for x.

14. Compute the value of x^3-3x^2+x+1, if $x=1-\sqrt{2}$.

15. Show that $x=-1\pm\dfrac{\sqrt{2}}{2}$ satisfies the equation

$$2x^3-7x-2=0.$$

16. Show that each of the four numbers $\pm\sqrt{3}\pm\sqrt{2}$, when substituted for x, will satisfy the equation

$$x^4-10x^2+1=0.$$

17. Compute the value of

$$9x(3x-2)+2,$$

if

$$x=\frac{1}{3}\left(1\pm\frac{1}{\sqrt{3}}\right).$$

18. Show that

$$x^3+13x^2-112x+98=0,$$

if

$$x=-7\left(1+\sqrt{3}\right).$$

Simplify the following and express the result with positive exponents:

19. $\left(x^{-2}\sqrt{x^3\sqrt{4}}\right)^4$.

20. $\left(\sqrt[3]{-27x^9}\right)^{-2}$.

21. $\left(x^{-2}\sqrt{x^3\sqrt[3]{x^2}}\right)^{-\frac{1}{2}}$.

22. $(ab^{-2}c^3)^{\frac{1}{2}}(a^6b^3c^{-2})^{\frac{1}{3}}$.

23. $\left(\dfrac{(ab)^{-3}}{a^{-2}b^2}\cdot\dfrac{a^{-\frac{1}{2}}b^{-\frac{2}{3}}}{a^{-6}b^{-1}}\right)^{-3}$.

24. $\dfrac{\left(\dfrac{a}{27}\div\dfrac{a^{-2}}{8}\right)^{-\frac{2}{3}}-x^2}{3a^{-1}+2x}$.

Change into an equivalent fraction with rational denominator:

25. $\dfrac{2a}{-b-\sqrt{b^2-4ac}}$.

26. $\dfrac{y^2}{x+\sqrt{x^2-y^2}}$.

27. $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

28. $\dfrac{\sqrt{x}+\sqrt{x+y}}{\sqrt{x}+\sqrt{y}+\sqrt{x+y}}$.

Change into an equivalent fraction with rational numerator:

29. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$.

30. $\dfrac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$.

31. $\dfrac{x+2-\sqrt{x^2-4}}{x+2+\sqrt{x^2-4}}$.

Find to three decimal places the value of:

32. $\dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$.

33. $\dfrac{15+7\sqrt{3}}{9+5\sqrt{3}}$.

34. $\dfrac{3}{3-2\sqrt{3}}-\dfrac{3}{3+2\sqrt{3}}$.

35. $\sqrt[n]{\dfrac{16}{2^{4+n}}}$.

36. $\dfrac{13\sqrt{15}-7\sqrt{21}}{13\sqrt{1\frac{2}{3}}-7\sqrt{2\frac{1}{3}}}$.

37. $\dfrac{\sqrt{\frac{3}{2}}-\sqrt{\frac{8}{27}}}{\sqrt{2}\left(\sqrt{3}+\dfrac{1}{\sqrt{3}}\right)}$.

38. $\sqrt{\dfrac{\dfrac{1}{2}\sqrt{3}-\dfrac{1}{\sqrt{2}}}{\dfrac{1}{2}\sqrt{3}+\dfrac{1}{\sqrt{2}}}}$.

39. $\sqrt[3]{4a^{-\frac{2}{5}}+b^0\sqrt{ab^{-1}}}$, when $a=-32$, $b=-8$.

40. $\sqrt{\frac{3}{2}p^{\frac{2}{3}}-\sqrt[3]{rs^{-2}t}}$, when $p=8$, $r=3$, $s=-1$, $t=9$.

41. $\dfrac{\sqrt{7}+\sqrt{5}-\sqrt{2}}{\sqrt{7}+\sqrt{5}+\sqrt{2}}$.

6. The Binomial Theorem. The formula for the expansion of any positive integral power of the binomial expression $a+b$ is as follows:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n. \qquad (1)$$

From this expansion the following rule for writing down the successive terms of a particular expansion may be deduced.

The first term is a^n *and the last is* b^n.

The second term is $na^{n-1}b$.

The exponents of a *decrease by* 1 *in each term after the first.*

The exponents of b *increase by* 1 *in each term after the second.*

The product of the coefficient of any term and the exponent of a *in that term, divided by the exponent of* b *increased by* 1, *gives the coefficient of the next term.*

The sign of each term of the expansion is + *if* a *and* b *are positive; the signs of the even-numbered terms are* − *if* b *only is negative.*

The number of terms in the expansion is $n+1$.

This formula can be established in elementary algebra only when n is a positive integer. But the same form of expansion, except for the last term, is valid when the exponent is fractional or negative, provided b is numerically less than a. This condition is necessary in order that the resulting series, which has an infinite number of terms, may have a meaning. With the restriction noted, we shall assume the validity of expansion (1) for any rational value of n without attempting to give a proof. A rigorous demonstration demands a knowledge of the calculus.

EXERCISES

Expand Exercises 1–3 completely; 4–8 to five terms:

1. $(2x+y)^4$.

2. $(1-x\sqrt{2})^5$.

3. $\left(\frac{a}{2}+\frac{b}{3}\right)^8$.

4. $(1+x)^{-2}$, $|x|<1$.

NOTE. By $|x|$ is meant the numerical value of x regarding the sign as positive. Thus $|-2| = +2$. $|x|$ is called the **absolute value** of x.

5. $\left(a^{-\frac{1}{6}}+\frac{b^2}{2}\right)^{-4}$.

6. $\dfrac{1}{x^{-2}+3}$.

7. $\left(\frac{2}{x}+1\right)^{\frac{3}{2}}$.

8. $\left(1-\frac{2x}{y}\right)^{-\frac{2}{3}}$.

9. Obtain the expansion $(1-\frac{2}{5})^{\frac{5}{2}} = \frac{3}{10} - \frac{1}{50} - \frac{1}{1000} - \cdots$.

10. Show also that $(1-\frac{2}{5})^{\frac{5}{2}} = \frac{9}{125}\sqrt{15}$.

11. Show from each of the above results that an approximate value of $(1-\frac{2}{5})^{\frac{5}{2}}$ is .279.

7. Ratio and proportion. The **ratio** of one number to a second number is the fraction found by dividing the first number by the second.

The ratio of a to b is denoted by $a : b$ or by $\frac{a}{b}$.

The dividend in this implied division is called the **antecedent**; the divisor is called the **consequent.**

Four numbers, a, b, c, d, are in **proportion** when the ratio of the first pair equals the ratio of the second pair.

This is denoted by $a : b = c : d$ or by $\frac{a}{b} = \frac{c}{d}$.

The letters a and d are called the **extremes,** b and c the **means,** of the proportion.

If a, b, c, d are in proportion, that is, if

$$a : b = c : d, \tag{I}$$

then

$$ad = bc, \tag{II}$$

$$b : a = d : c, \tag{III}$$

$$a : c = b : d, \tag{IV}$$

$$a + b : b = c + d : d, \tag{V}$$

$$a - b : b = c - d : d, \tag{VI}$$

$$a + b : a - b = c + d : c - d. \tag{VII}$$

Equation (III) is said to be derived from (I) by **inversion.**

Equation (IV) is said to be derived from (I) by **alternation.**

Equation (V) is said to be derived from (I) by **addition.**

Equation (VI) is said to be derived from (I) by **subtraction.**

Equation (VII) is said to be derived from (I) by **addition and subtraction.**

8. Variation. The number x is said to **vary directly** as the number y when the ratio of x to y is constant. This we symbolize by

$$x \propto y, \text{ or } \frac{x}{y} = k,$$

where k is a constant.

The number x is said to **vary inversely** as the number y when x varies directly as the reciprocal of y. Thus x varies inversely as y when

$$x \propto \frac{1}{y}, \quad \text{or} \quad \frac{x}{\frac{1}{y}} = xy = k,$$

where k is a constant.

The intensity of a light varies inversely as the square of the distance of the light from the point of observation. If l represents the intensity of the light and d its distance from the point of observation, we have

$$l \propto \frac{1}{d^2}, \quad \text{or} \quad \frac{l}{\frac{1}{d^2}} = ld^2 = k,$$

where k is a constant.

The number x is said to **vary jointly** as y and z when it varies directly as the product of y and z. Thus x varies jointly as y and z when

$$x \propto yz, \quad \text{or} \quad \frac{x}{yz} = k,$$

where k is a constant.

EXERCISES

1. Prove that if $a : b = c : d = e : f$, then

$$\frac{ka + lc - me}{kb + ld - mf} = \frac{a}{b}.$$

2. Prove that if $a : b = c : d$, then

$$\frac{a^2 + b^2}{ab} = \frac{c^2 + d^2}{cd}.$$

3. The surfaces of similar solids have the same ratio as the squares of their corresponding dimensions, and their volumes have the same ratio as the cubes of their corresponding dimensions. What is the ratio of the surfaces of two cubes if the volume of one is twice that of the other? What is the ratio of the volumes of two spheres if the surface of one is twice that of the other?

4. If $x \propto y$, and $x = 6$ when $y = 10$, find y when $x = 15$.

5. If $x \propto \frac{1}{y}$, and $x = 4$ when $y = 100$, find x when $y = 10$.

6. If $x \propto yz$, and $x = 3$ when $y = 4$ and $z = 5$, find x when $y = 20$ and $z = 2$.

7. A man 6 feet tall is walking directly away from a lamp-post 10 feet high. Find the ratio of the length of his shadow on the ground, to the distance of the further end of his shadow from the lamp-post. How long is the man's shadow when he is 20 feet from the lamp-post?

8. The safe load of a horizontal beam supported at both ends varies jointly as the breadth and the square of the depth, and inversely as the length between supports. If a 3 by 9 inch beam 15 feet long, standing on edge, safely supports a weight of 1800 pounds, find the safe load of a $2\frac{1}{2}$ by 6 inch beam of the same material 8 feet long.

9. The weight of a body above the surface of the earth varies inversely as the square of its distance from the center of the earth, and its weight below the surface varies directly as its distance from the center. A body weighs 100 pounds at the surface of the earth. What would it weigh 1000 miles above the surface? 1000 miles below the surface? (Radius of the earth = 4000 miles.)

10. A disk 1 foot in diameter held 1.2 feet from the eye just obscures a ball whose center is 13 feet from the eye. If the ball is moved away so that the distance of its center from the eye is 25 feet, how far from the eye must the disk be held so that the ball is just obscured?

11. If $x^{-\frac{2}{3}} : 2 = 1 : x^{\frac{1}{6}}$, what is the value of x?

12. A and B are 6 and 16 years old respectively. In how many years will the ratio of their ages be $2:3$?

13. The time required by a pendulum to make one vibration varies as the square root of its length. If a pendulum 100 centimeters long vibrates once in 1 second, find the time of one vibration of a pendulum 81 centimeters long. What is the length of a pendulum which vibrates once in 2 seconds?

14. The volume of a cylinder varies jointly as its altitude and the square of its diameter. The diameter of two cylinders are in the ratio $3:2$, and the volume of the second is two fifths that of the first. Find the ratio of their altitudes.

15. Kepler's third law of planetary motion states that the square of a planet's time of revolution varies as the cube of its mean distance from the sun. The mean distances of the earth and Mercury from the sun are 93 and 36 millions of miles respectively. Find the time of Mercury's revolution.

16. The electric resistance of a wire varies directly as its length and inversely as the square of its diameter. Its weight varies jointly as its length and the square of its diameter. What must be the length and diameter of a wire which is to have double the resistance but only two fifths the weight of a wire of the same material 100 feet long and .02 inch in diameter?

9. Arithmetical progression. An **arithmetical progression** is a succession of terms in which each term after the first, minus the preceding one, gives the same number.

This number is called the **common difference** and may be positive or negative.

The formulas for the nth term, t_n, and for the sum of n terms, S_n, respectively, are as follows:

$$t_n = a + (n-1)d,$$

$$S_n = \frac{n}{2}(a + t_n),$$

where a is the first term, n the number of terms, and d the common difference.

10. Geometrical progression. A **geometrical progression** is a succession of terms in which each term after the first, divided by the preceding one, always gives the same number. This constant quotient is called the **ratio.**

The formulas for the nth term, t_n, and for the sum of the first n terms, S_n, respectively, are as follows:

$$t_n = ar^{n-1},$$

$$S_n = \frac{a - ar^n}{1 - r},$$

where a is the first term, n the number of terms, and r the ratio.

When r is numerically less than 1, the successive terms of a geometrical series become numerically less and less, and the sum of n terms approaches a fixed number as a limit as n increases indefinitely. This limit is called the sum of the infinite geometrical series, and is given by the formula

$$S_\infty = \frac{a}{1 - r}.$$

This formula must never be used when r is greater than unity, for in that case the corresponding series does not approach a limit.

EXERCISES

1. Find the 10th term and the sum of the first 10 terms of the progression $1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$.

2. Find the $(n-2)$d term and the sum of the first $n-1$ terms of the progression $a, a+d, a+2d, \cdots$.

3. Find the 8th term and the sum of the first 8 terms of the progression $2, 3, \frac{9}{2}, \cdots$.

4. Find the sum of the infinite series $3 - 1 + \frac{1}{3} - \cdots$.

5. (a) Find the sum of the infinite series

$$1 + \frac{1}{x} + \frac{1}{x^2} + \cdots. \qquad (x > 1)$$

(b) Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

6. A body starting from rest falls 16 feet the first second, 48 the next, 80 the next, and so on. How far does it fall during the 10th second? How far has it fallen at the end of the 10th second?

7. Using the information given in the preceding exercise, deduce a general formula for the distance that a body will fall in t seconds.

8. A man standing on a cliff wishes to determine his height above its foot. He drops a stone and notices that it strikes the ground in 4 seconds. How high is the cliff?

9. The first term of a geometrical progression is 225 and the fourth term is $14\frac{2}{5}$. Find the series and sum it to infinity.

10. Twelve potatoes are placed in line at distances 6, 12, 18, $\cdots$ feet from a basket. A boy, starting from the basket, picks up the potatoes and carries them back one at a time to the basket. How far must he run to complete the potato race?

11. How far must a boy run in a potato race if there are n potatoes at a distance d feet apart, the first being at a distance a feet from the basket?

12. A chain letter is written, each person receiving the letter rewriting it and sending it to two others. If the first person sends out two letters, how many letters will have been written after all the tenth letters of the chain have been sent?

13. A chain letter is written, each person in the chain sending out a letters. How many letters will have been written after all the nth letters of the chain have been sent?

11. Linear equations in one variable. The following definitions may be found useful for reference.

An **equation** is a statement of equality between two equal numbers or number symbols.

Equations are of two kinds — **identities** and **equations of condition.**

An arithmetical or an algebraic identity is an equation in which, if the indicated operations are performed, the two members become precisely alike, term for term.

For example,

$$(a-b)^2 = a^2 - 2\,ab + b^2, \quad a \cdot \frac{b}{a} - b = 0, \quad \text{and} \quad 2^3 - 3 \cdot 2^2 - 4 \cdot 2 + 12 = 0,$$

are identities.

A literal identity is true for any value of the letters involved.

An equation which is true only for certain values of the letter or letters involved is an equation of condition, or simply an equation.

For example, $x - 2 = 0$, $(x-1)(x+3) = 0$, and $x^2 - 1 = y^2$ are equations of condition.

A number or number symbol which being substituted for the unknown letter in an equation changes the equation to an identity is said to **satisfy** the equation.

After the substitution is made it is usually necessary to simplify the result before the identity becomes apparent.

A **root** of an equation is any number or number symbol which satisfies the equation.

We assume the following

AXIOM. *If equals be added to, subtracted from, multiplied by, or divided by equals, the results are equal.*

As always, we exclude division by zero. In dividing each member of an equation by an algebraic expression one must note for what values of the letters the divisor vanishes and exclude these values from the discussion.

An equation is solved when its roots have been found. The process of solution is the application of the foregoing axiom to the equation in such a manner as to obtain the unknown alone in one member of the equation.

Suppose the equation $4\,x - 7 = 17$ is given, and the numerical value of x is wanted. The validity of the equation is not affected if 7 is added to each member; that is, if we write $4\,x = 24$, x remains the same number as in the original equation. The same

may be said if we divide each side of the new equation by 4 and obtain $x = 6$. It is of the greatest importance to understand that when we perform the operations mentioned in the axiom, we are not getting anything new, but are expressing in a more available form some symbol whose precise value did not appear clearly from the original equation. In the illustration just used x is no more truly equal to 6 after our process of solution than it was before; it *must* be equal to 6 if this particular statement is an equation. We have merely rewritten the equation $4x - 7 = 17$ so that x appears alone in one member. From this point of view the fact that the number which one obtains as a result of solving an equation satisfies the equation should not be surprising. If the work has been correctly done, it is impossible that it should be otherwise, for the unknown is the same number at the end of the solution that it was at the beginning.

We may say, then, that the root of an equation is obtained by modifying the form of the original equation so as to display the value of the letter, which for the time being is unknown, in terms of what is numerically known. It cannot be too strongly insisted that the solution of an equation consists in finding its roots, and that the only property of a root of an equation which distinguishes it from other numbers is that it satisfies its equation. If in the hypothesis of a theorem a number is given as the root of a certain equation, we know that if the number be substituted for the unknown, the resulting equation is an identity.

When we change an equation from one form to another in the process of solving, it is assumed that x can have no value which would reduce to zero the denominator of any fraction appearing in the process.

EXAMPLES

1. Solve the equation

$$\frac{2x}{3}\left(1+\frac{5}{x}\right)+\frac{3x}{4}\left(1+\frac{4}{x}\right)=0.$$

Solution. Here we assume that x cannot equal 0, for this would cause the denominators of two fractions to become 0.

Multiplying, we have

$$\frac{2x}{3}+\frac{10x}{3x}+\frac{3x}{4}+\frac{12x}{4x}=0.$$

Since x is not 0, we can divide it out of both the numerator and denominator of the second fraction and the fourth fraction. This gives

$$\frac{2x}{3}+\frac{10}{3}+\frac{3x}{4}+3=0.$$

$$8x+40+9x+36=0.$$

$$17x=-76.$$

$$x=-\tfrac{76}{17}.$$

Check.

$$\frac{2(-\frac{76}{17})}{3}\left(1+\frac{5}{-\frac{76}{17}}\right)+\frac{3(-\frac{76}{17})}{4}\left(1+\frac{4}{-\frac{76}{17}}\right)$$

$$=-\tfrac{152}{51}\cdot\left(-\tfrac{9}{76}\right)-\tfrac{228}{68}\cdot\tfrac{8}{76}$$

$$=\tfrac{6}{17}-\tfrac{6}{17}=0.$$

2. Solve the equation

$$\frac{1}{a+b}+\frac{a+b}{x}=\frac{1}{a-b}+\frac{a-b}{x}.$$

Solution. We assume that $a+b\neq 0,\ a-b\neq 0,\ x\neq 0.$

Clearing of fractions,

$$ax-bx+a^3-ab^2+a^2b-b^3=ax+bx+a^3-ab^2-a^2b+b^3.$$

Transposing,

$$-2bx=-2a^2b+2b^3.$$

$$x=a^2-b^2.$$

In taking this step we have assumed that $b\neq 0$.

Check. Substituting a^2-b^2 for x in the original equation, it becomes

$$\frac{1}{a+b}+\frac{a+b}{a^2-b^2}=\frac{1}{a-b}+\frac{a-b}{a^2-b^2},$$

which, under the assumption that $a+b\neq 0,\ a-b\neq 0,$ reduces to

$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{1}{a-b}+\frac{1}{a+b},$$

an identity. Hence a^2-b^2 is a root if $b\neq 0$.

In the course of the foregoing solution, the value $b=0$ was excluded. The equation must now be considered for this special case. If $b=0$, the original equation becomes

$$\frac{1}{a}+\frac{a}{x}=\frac{1}{a}+\frac{a}{x},$$

which is an identity. Hence x is indeterminate.

Shorter solution of example 2:

Transposing,

$$\frac{a+b}{x}-\frac{a-b}{x}=\frac{1}{a-b}-\frac{1}{a+b},$$

$$\frac{2b}{x}=\frac{2b}{a^2-b^2},$$

$$x=a^2-b^2,$$

with the same restrictions on the letters as before.

EXERCISES

Solve the following equations and in each case verify the fact that the result satisfies the equation:

1. $\dfrac{7x}{3} - \dfrac{1}{6} = \dfrac{9x}{4}.$

2. $\dfrac{5x-7}{6} - \dfrac{5}{2}\left(\dfrac{4-x}{10}\right) = \dfrac{15x-22}{6}.$

3. $.5704 - .20x = 19.6512 - .016x.$

4. $\dfrac{cy}{2d} - c^2 = \dfrac{2dy - c^3}{c}.$

5. $\dfrac{1}{2}\left(x - \dfrac{a}{3}\right) - \dfrac{1}{3}\left(x - \dfrac{a}{4}\right) + \dfrac{1}{4}\left(x - \dfrac{a}{5}\right) = 0.$

6. $\dfrac{3abc}{a+b} + \dfrac{a^2b^2}{(a+b)^3} + \dfrac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \dfrac{bx}{a}.$

7. $1.2x - \dfrac{.18x - .05}{.5} = .4x + 8.9.$

8. $\dfrac{x}{4}(x-1) - \left(\dfrac{x+1}{2}\right)^2 = \dfrac{2}{3}\left(x - \dfrac{1}{2}\right).$

9. $0 = \dfrac{2x}{3} - \dfrac{x}{2} + \dfrac{x}{6} - 1\frac{3}{4}x + 2\frac{4}{5} - 3\frac{9}{10}x + 61.$

10. $2x - 3 = 2.25x - 5 - .4x + 2.6.$

11. $.5555 = 5.55x + 333.33 - 44.4x - 30.91.$

12. $0 = 2x - 3(5 + \frac{3}{4}x) + \frac{2}{3}(4 - x) - \frac{1}{4}(3x - 16).$

13. $11 - \left(\dfrac{3x-1}{4} + \dfrac{2x+1}{3}\right) = 10 - \left(\dfrac{2x-5}{3} + \dfrac{7x-1}{8}\right).$

14. $\frac{2}{7}\{\frac{5}{12}[\frac{7}{8}(\frac{3}{4}x + 5) - 10] + 3\} - 8 = 0.$

15. $7\frac{1}{3}x - 2\frac{1}{2} - [4\frac{1}{2} - \frac{1}{2}(3\frac{1}{3} - 5x)] = 18\frac{1}{3}.$

16. $\dfrac{7}{3} - \dfrac{13x - 24}{3x} = \dfrac{10}{x} - \dfrac{37}{20} - \dfrac{13}{5x}.$

17. $\dfrac{\frac{3}{4}(5x+1)}{\frac{2}{3}(4x-1)} = \dfrac{3}{2}.$

18. $\dfrac{1}{x + \frac{1}{3}} = 3 - \dfrac{1}{3}.$

19. $\dfrac{1.3 - 3x}{2} = \dfrac{1.8 - 8x}{1.2} - \dfrac{5x - .4}{.3}.$

20. $\frac{3}{4}\cdot\frac{4x-5}{3x-7}=\frac{5}{7}\cdot\frac{7x-3}{5x-4}$.

21. $6(x-6)=\frac{3x-14}{x-4}(2x-11)$.

22. $ab-(x-c)d=c(d+x)$.

23. $m(a+b-x)=n(a+b-x)$.

24. $(a+c-x)(a+b)+(a-c+x)(a-b)=2a^2$.

25. $(a-b)(a-c)(a+x)+(a+b)(a+c)(a-x)=0$.

26. $\frac{ax}{b}+\frac{bx}{a}+\frac{2ab}{a+b}=\frac{(a+b)^2x}{ab}$.

27. $\frac{a-x}{a}+\frac{b-x}{b}+\frac{c-x}{c}=3$.

28. $\frac{1}{3}[\frac{1}{3}(\frac{1}{3}(\frac{1}{3}(\frac{1}{3}x+2)+2)+2)+2]=1$.

29. $\frac{1}{2}[\frac{1}{2}(\frac{1}{2}(\frac{1}{2}(\frac{1}{2}x-1)-1)-1)-1]=1$.

30. $\frac{\frac{2}{3}x-\frac{2}{3}}{\frac{2}{3}-x}-\frac{2}{3}=\frac{2}{3}-\frac{\frac{2}{3}x+\frac{2}{3}}{\frac{2}{3}-x}$.

31. $\frac{a-\frac{1}{x}}{a+\frac{1}{x}}-\frac{1}{x}=\frac{x-\frac{1}{a}}{x+\frac{1}{a}}-\frac{1}{a}$.

32. $\frac{9}{x-51}-\frac{9}{x-15}=\frac{2}{x-81}-\frac{2}{x+81}$.

33. $\frac{3x-7}{2x-9}-\frac{3(x+1)}{2(x+3)}=\frac{11x+3}{2x^2-3x-27}$.

34. $\frac{x-a}{x-m}+\frac{x-b}{x-n}=2$.

35. $\frac{a^2b-x}{a}+\frac{b^2c-x}{b}+\frac{c^2a-x}{c}=0$.

36. $\frac{a+1}{x}:\frac{b-1}{x}=(a+x):(b-x)$.

37. $(x-a\sqrt{b}):(x-b\sqrt{a})=\sqrt{b}:\sqrt{a}$.

38. $\frac{ac}{b(a-b)m}-\frac{(m+n)^2x}{bm}-\frac{nx}{b}=\frac{c}{(a-b)m}-\frac{3nx}{b}$.

39. $\frac{a^2+b^2}{b}(x-a)+\frac{a^2-b^2}{a}(x-b)=2a(2a+b-x)$.

12. Linear equations in two variables. In the last section it was seen that any linear equation in one unknown may be solved; that is, its root may be found. The equation $ax + b = 0$, which is the most general form for the linear equation in one variable, has one and only one root, namely $x = -\frac{b}{a}$, for this is the only number which, when substituted for the unknown in such an equation, will satisfy it.

If we consider a linear equation in two unknowns, as, for example, $2x - 3y = 6$, it appears that for any particular value of y the equation becomes a linear equation in the single unknown x, and therefore has one root. Hence this equation has not only a single pair of values which satisfies it, but countless pairs.

For instance, if we let $y = 1$, the equation becomes $2x - 3 = 6$, which has the root $\frac{9}{2}$. If $y = 4$, x has the value 9.

If we have two linear equations, each in two unknowns, each is satisfied by countless pairs of numbers. The process of solving the system of two equations determines whether there is any pair of values of the unknowns which satisfies both equations simultaneously. For this reason the two equations may be called a **simultaneous system of equations.** Usually such a system has one and only one pair of common roots. Sometimes no such values exist. In the latter case the equations are called **incompatible.** If two equations become identical when each member of one of them is multiplied by some constant, they are called **dependent** equations. As a general thing pairs of equations are independent and compatible.

EXAMPLES

1. Solve $\begin{cases} 3x + y = 5, & (1) \\ 2x - 5y = 9. & (2) \end{cases}$

Solution. Multiplying (1) by 5 and adding (2),

$$\begin{aligned} 15x + 5y &= 25 \\ 2x - 5y &= 9 \\ \hline 17x \qquad &= 34 \\ x &= 2. \end{aligned}$$

Substituting in (1),

$$\begin{aligned} 6 + y &= 5, \\ y &= -1. \end{aligned}$$

Check. Substitute in (1) and (2), $6 - 1 = 5$, $4 + 5 = 9$.

2. Solve $\begin{cases} ax + by = (a^2 - b^2)c, & (1) \\ bx - ay = 2\,abc. & (2) \end{cases}$

Solution. Multiplying (1) by a, (2) by b, and adding,

$$\begin{aligned} a^2x + aby &= (a^2 - b^2)\,ac \\ b^2x - aby &= 2\,ab^2c \\ \hline (a^2 + b^2)\,x &= (a^2 + b^2)\,ac \\ x &= ac. \end{aligned}$$

Substituting in (2),

$$\begin{aligned} abc - ay &= 2\,abc, \\ -ay &= abc, \\ y &= -bc. \end{aligned}$$

Check. Substituting the values found for x and y in (1) and (2), we obtain

$$a^2c - b^2c = (a^2 - b^2)\,c,$$
$$abc + abc = 2\,abc.$$

EXERCISES

Solve the following systems of equations and check the results:

1. $4x + 2y = 0,$ $3x - y = 15.$

2. $8x - 3y + 16 = 0,$ $5y + 6x = 17.$

3. $11.3\,x + .125\,y = 1255,$ $10.3\,x - y = 30.$

4. $7x - 10y = .1,$ $11x - 16y = .1.$

5. $23x + 15y = 4\frac{1}{4},$ $48x + 45y = 18.$

6. $2\frac{1}{4}x = 3\frac{1}{3}y + 4,$ $2\frac{1}{5}y = 3\frac{1}{3}x - 47.$

7. $\frac{3}{4}x - \frac{1}{2}(y + 1) = 1,$ $\frac{1}{3}(x + 1) + \frac{3}{4}(y - 1) = 9.$

8. $3.5\,x + 2\frac{1}{3}y = 13 + 4\frac{1}{7}x - 3.5\,y,$ $2\frac{1}{7}x + .8\,y = 22\frac{1}{2} + .7\,x - 3\frac{1}{3}y.$

9. $x : y = 3 : 4,$ $(x - 1) : (y + 2) = 1 : 2.$

10. $(x + 4) : (y + 1) = 2 : 1,$ $(x + 2) : (y - 1) = 3 : 1.$

11. $(x + 1) : (y + 1) : (x + y) = 3 : 4 : 5.$

12. $(x - 2) : (x + 1) : (x + y - 3) = 3 : 4 : 5.$

13. $(x - 5) : (y + 9) : (x + y + 4) = 1 : 2 : 3.$

14. $(x + y - 4) : (2x + y + 1) = 1 : 2,$ $(2x + y - 9) : (x + 2y + 7) = 3 : 4.$

15. $\dfrac{x+1}{3} - \dfrac{y+2}{4} = \dfrac{2(x-y)}{5},$ $3(x - 3) - 4(y - 3) = 12(2y - x).$

16. $$\frac{3x+2y}{5}+\frac{5x+3y}{3}=x+1,\quad \frac{2x+3y}{3}+\frac{4x+3y}{2}=1-y.$$

17. $$\frac{15}{x}-\frac{4}{y}=4,\quad \frac{3}{x}+\frac{8}{y}=3.$$

HINT. Retain fractions.

18. $$\frac{x}{3}+\frac{5}{y}=4\tfrac{1}{3},\quad \frac{x}{6}+\frac{10}{y}=2\tfrac{2}{3}.$$

19. $$17x-\frac{.3}{y}=3,\quad 15x-\frac{.4}{y}=2.$$

20. $$\frac{1.6}{x}+\frac{2.7}{y}+1=0,\quad \frac{.8}{x}-\frac{3.6}{y}-5=0.$$

21. $$ax+by=c.\quad mx=ny.$$

22. $$ax+by=(a^2-b^2)c,\quad (a+b)x+(a-b)y=(a^2+b^2)c.$$

23. $$(a^2-b^2)(x+y)=a^2+b^2,\quad (a^2-b^2)(2x+3y)=2a^2+ab+3b^2.$$

24. $$\frac{x}{a+b}-\frac{y}{a-b}=\frac{2}{a+b},\quad \frac{x}{a+b}+\frac{y}{a-b}=\frac{2}{a-b}.$$

25. $$\frac{a}{x}-\frac{b}{y}+1=0,\quad \frac{b}{x}-\frac{a}{y}+1=0.$$

26. $$\frac{a}{a-x}+\frac{b}{b-y}=\frac{a}{b},\quad \frac{b}{a-x}+\frac{a}{b-y}=\frac{b}{a}.$$

27. $$(x+y):(x-y)=a:(b-c),\quad (x+c):(y+b)=(a+b):(a+c).$$

28. $$x:y=(a^3-b^3):(a^3+b^3),\quad (x-a):(y-a)=(a-b):(a+b).$$

29. $$2\sqrt{x+5}-3\sqrt{y-2}=3,\quad 3\sqrt{x+5}-4\sqrt{y-2}=5.$$

30. $$\frac{4}{\sqrt{x-3}}+\frac{9}{\sqrt{y+3}}=4,\quad \frac{8}{\sqrt{x-3}}-\frac{3}{\sqrt{y+3}}=1.$$

31. Show that the following equations are dependent:

$$4\tfrac{1}{2}x-3.75y=2.25,$$
$$12x-10y-6=0.$$

How many pairs of values of x and y satisfy both equations? Find two pairs of values of x and y that satisfy both equations.

32. Show that the following equations are dependent and find two pairs of values of x and y that satisfy them:

$$2.125\,x + 8\tfrac{1}{2} = .25\,y,$$
$$x + y + 4 = 1\tfrac{2}{17}\,y.$$

33. Show that the following equations are incompatible:

$$13\,x = 8 - 39\,y,$$
$$6\,y = 19 - 2\,x.$$

How many pairs of values of x and y satisfy both equations?

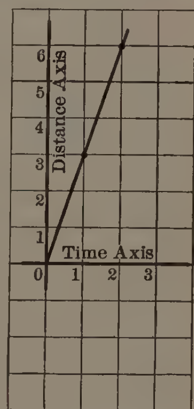
34. Show that the following equations are incompatible:

$$2.2\,x + 3\tfrac{2}{5}\,y = 2,$$
$$5(3\,x + 4\,y - 3) = 4\,x + 3\,y.$$

CHAPTER II

FUNCTIONS AND THEIR GRAPHS

13. Uniform motion. Suppose a man who is taking a long walk finds that at the end of one hour he has covered three miles, and that at the end of each successive hour he has advanced three additional miles on his way. We might represent the relation between the distance which he goes and the time it takes him to do it, by means of a graph as follows: Mark off on a horizontal line equal segments, each one of which represents an hour. On a line at right angles to this first line mark off equal segments, each one of which represents a mile, the point which represents zero hours and zero miles being at the intersection of the two lines. These lines we call the time axis and the distance axis respectively. To represent the fact that the man has walked three miles during the first hour, we make a dot just over the one-hour point, and three distance units above it. If he had gone only two miles in this first hour, we would have made the dot only two units above the time axis. Similarly, at the two-hour point we make a dot six distance units above the time axis, and so on for the succeeding hours. If the rate of walking was the same during the entire time, at the end of the first half hour he would have covered one and one-half miles, which could be represented by a dot over the mid-point of the first hour segment, one and one-half distance units from the time axis. We could insert in a similar manner as many other points as we might desire. It is evident that all of these dots lie on a straight line.



Now the relation between the distances of any point on this line from the two axes represents the relation between the distance which the man walks and the time it requires. If we wish to determine how far he had gone at the end, say, of one and a half hours, we

have only to observe how many units of distance above the time axis the line is at the point midway between the one- and the two-hour points. In fact, the figure represents graphically the law which tells us how far the man will have walked at the end of any number of hours.

The graph is not the only means we have of representing this law. We may use an equation for the same purpose. If we represent by s the number of miles he walks in t hours, the relation which we expressed by a graph above we may represent by the equation $s = 3\,t$. By means of this equation we can find out how far the man has walked in any number of hours, say, two and a half, by replacing the letter t in the equation by this number, and computing the value of s.

At first sight it might appear that the graphical method of representing the foregoing problem is less satisfactory than the other method, on account of the unavoidable inaccuracy in drawing the lines of the figure. For instance, it would be impossible to tell from the figure the distance covered in a certain time, correct to a foot, or even to a rod. It is to be observed, however, that it is equally impossible to measure the rate of walking with perfect accuracy, and although we say that the rate is three miles an hour, this is only approximate. It is a principle of great importance in applied mathematics that one cannot obtain by the use of formulas results which are more accurate than the data from which the formulas are derived. Consequently, in dealing with problems like the one just considered, if the drawing is carefully done, results as accurate as the original measurement of the man's rate of walking can be obtained from the graph.

It is often convenient to use a different scale of measurement on the two axes, but this affects at most the shape of the graph obtained, and not the nature of the numerical relation which is represented.

14. The notion of function. The word **quantity** denotes anything which may be measured. Distance, weight, time, volume, surface, pressure, force, are all quantities, since each is measurable in terms of a suitable unit. When two quantities are so related to each other that when the first is given the second is determined, the second is said to be a **function** of the first.

It is unnecessary that there should be any causal relation between the quantities; the mere correspondence of values is sufficient to establish the functional relation. For example, the temperature on a given day depends

physically on the atmospheric conditions, the angle at which the rays of the sun strike the earth, and various other conditions. It does not depend causally on the time of day. But nevertheless, since to each time of day there corresponds a certain temperature, we may properly say that the temperature is a function of the time. In § 13 the distance which the man walks is a function of the time it takes him to do it. In applied mathematics, wherever there is motion or change or growth a functional relation exists.

It is a matter of importance to devise simple means of representing these functions, so that they may be studied and further relations discovered. The function mentioned in the last section was represented by two means — by a graph and by an equation. Each method was effective in displaying the fact and nature of the relation between the distance and the time, but they did it in quite different ways. The present text will concern itself with the study of these two methods of representing functions.

In what follows, the equation or the graph will often be studied entirely apart from any physical meaning which the letters or the lines may have. But it should never be forgotten that x and y in any equation may be the measures of physical quantities which it is desirable to determine.

An algebraic expression involving the letter x is a function of x because, corresponding to the various values of x, one or more values of the expression can be determined.

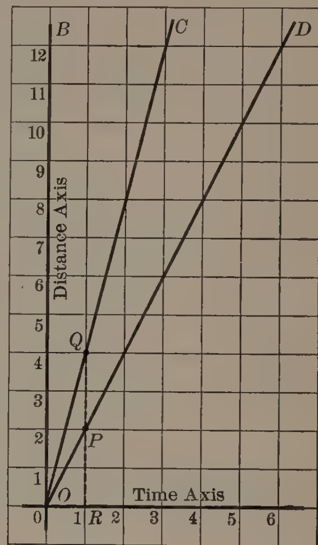
Thus $2x^2+1$, $\sqrt{x-2x^4}$, and $\dfrac{2x+3x^3}{x+1}$ are each functions of x.

Expressions involving two or more letters are called functions of those letters. By means of equations involving such algebraic expressions, numerical relations between the letters are defined. Thus the equation $x=2y^2$ tells us that x and y are so related that x always equals twice the square of y.

EXAMPLES

1. Two towns, A and B, are 12 miles apart. A man walks from A to B at the rate of 2 miles an hour. Express the distance s which he travels, as a function of the time t, and represent the function graphically. Determine from the graph what time will be required for him to reach B. If he travels at the rate of 4 miles an hour, what time will be required for him to reach B? What is the relation between the angles which the lines representing the functions in the two cases make with the time axis?

Solution. Draw a pair of axes, the time axis and the distance axis, at right angles, and let O, their point of intersection, represent the first point where $t = 0$ and $s = 0$. Since the man's first rate is 2 miles an hour, we make a dot, P, at a distance of 2 units above the point on the time axis where t equals one unit. Through P and O draw a straight line. This line is the graph of the function $s = 2t$.



Now we wish to find what value t has when $s = 12$, that is, how long will be required for the man to walk 12 miles. Through the point B, 12 units up on the distance axis, we draw a parallel to the time axis. This line contains all points for which $s = 12$. Let D be the point where this line intersects the graph of the function $s = 2t$. Dropping a perpendicular from D to the time axis, we see that for the point D, $t = 6$, that is, when $s = 12$, $t = 6$; that is, 6 hours are required to walk a distance of 12 miles.

Similarly, if the man's rate is 4 miles an hour, we make a dot, Q, 4 units above the point on the time axis where $t = 1$, and draw a line through Q and O. Let this line intersect B and D at C. Corresponding to the point C, $t = 3$; that is, 3 hours are required to go 12 miles at the second rate. The function in this case is $s = 4t$.

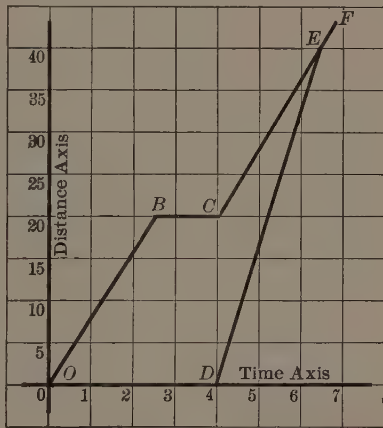
We notice that if the rate of the man is increased, the graph of the function becomes steeper. Doubling the rate did not double the angle POR, since $\angle QOR$ is not equal to $2\angle POR$, but the ratio $\frac{RQ}{OR}$ equals twice the ratio $\frac{RP}{OR}$. The ratio $\frac{RP}{OR}$ is called the **slope** of the line OD, and $\frac{RQ}{OR}$ is the slope of the line OC. The slope of the line is the rate at which the man is traveling. Doubling the rate, then, doubles the slope of the line representing the function.

2. A man starts out to ride on a bicycle at the rate of 8 miles an hour. After riding $2\frac{1}{2}$ hours he stops for $1\frac{1}{2}$ hours, then continues at his former rate. Four hours after the first man starts, a second man leaves the same place on a motor cycle at the rate of 16 miles an hour. How far must the second man ride to overtake the first?

Graphical solution. Let O be the intersection of the time axis and the distance axis; that is, the point where $t = 0$ and $s = 0$. Draw a line through O whose slope is 8. This line is the graph which represents the relation between s and t for the first $2\frac{1}{2}$ hours. Now for the next $1\frac{1}{2}$ hours s does not increase but remains the

same. This is denoted by a line BC parallel to the t axis and $1\frac{1}{2}$ units long. The man now continues at his former rate. This is denoted by a line CF having the same slope as the line OB.

The second man starts 4 hours later; that is, when $t = 4$, but since he starts from the same place as the first man, $s = 0$. This is denoted by the point D. Through D draw a line whose slope is 16. This line is the graph of the function which represents the motion of the second man. The two graphs intersect at a point E for which $s = 40$. Hence the second man overtakes the first after riding 40 miles.



It should be noticed that the lines $OBCE$ and DE are not the paths of the two men but the graphs which represent the relation between the distance traveled and the time for the first and second men respectively. The point E is not the intersection of the paths of the men, but the point on the two graphs where s has the same value.

Algebraic solution. Let x be the distance in miles which the second man must ride in order to overtake the first. In $2\frac{1}{2}$ hours, at 8 miles an hour, the first man rides 20 miles. Then $x - 20 =$ the distance the first man rides after the delay, and the time required to ride this distance was $\frac{x-20}{8}$ hours. The total time of the first man is $2\frac{1}{2} + 1\frac{1}{2} + \frac{x-20}{8}$ hours. The time of the second man is $4 + \frac{x}{16}$ hours. These times are equal. Hence the equation

$$2\tfrac{1}{2} + 1\tfrac{1}{2} + \frac{x-20}{8} = 4 + \frac{x}{16},$$

$$\frac{x-20}{8} = \frac{x}{16},$$

$$2x - 40 = x,$$

$$x = 40.$$

EXERCISES

Solve the following exercises both graphically and algebraically:

1. Two men start at the same time to walk a distance of 15 miles, the first at 3 miles an hour, the second at $2\frac{1}{2}$ miles an hour. How much sooner will the first arrive than the second?

2. A man walking 2 miles an hour leaves a town A. He is followed by a second man who leaves A 4 hours later, walking 4 miles an hour. How long must the second man walk to overtake the first?

3. A man starts out to walk at a uniform rate and finds that at the end of 2 hours he has walked 7 miles. If he continues at the same rate for 3 hours longer, how far will he have walked?

4. A stone is dropped into a pond and sends out a series of ripples. If the radius of the outer ripple increases at the rate of 5 feet a second, what is the length of the circumference of the outer ripple at the end of 3 seconds?

5. Two towns, A and B, are 44 miles apart. A man leaves A for B at 8 A.M., riding a bicycle at a uniform rate. At 9.30 an accident detains him for 30 minutes at a point 12 miles from A, after which he doubles his rate. At what time will he reach B?

6. A trip of 90 miles was made in an automobile in 5 hours. The first part of the trip was made at a uniform rate of 15 miles an hour and the last part at 20 miles an hour. How much of the distance was run at the latter rate?

7. A man walks 5 miles in $2\frac{1}{2}$ hours, then 12 miles further in 3 hours. What uniform rate would he have taken to cover the same distance in the same time?

8. A man starts out to walk at the rate of 3 miles an hour, and after walking for $1\frac{1}{2}$ hours he rests half an hour and then continues walking at the same rate. Another man leaving the same place 4 hours later on a bicycle, rides at the rate of 12 miles an hour. How far must he ride to overtake the first man?

9. Two piers, A and B, are on opposite sides of a lake 12 miles wide. A boat leaves A, crossing the lake at the rate of 12 miles an hour. Thirty minutes later another boat starts from B to A, making 18 miles an hour. How far from pier A will the boats pass each other?

10. Two automobiles are running in the same direction around a circular track. They make the circuit in 1 minute 30 seconds and 2 minutes 15 seconds respectively. If they start together, after how many minutes will they be together again?

11. A tank has two outlet pipes. By one it can be emptied in 12 minutes and by the other it can be emptied in 4 minutes. If both pipes are opened, find the number of minutes required to empty the tank.

12. Two automobiles leave a certain place at the same time running in opposite directions, the first at 16 miles per hour and

the second at 28 miles per hour. After going a certain distance the second turns around, continues at the same rate, and overtakes the first an hour and a half after the start. How far does the second car go before turning around?

15. Dependent and independent variables. Consider the equation which expresses the relation between the area of a circle and its radius, $A = \pi r^2$. Of the two variables A and r, one, the radius, can usually be measured and the corresponding value of A determined. In this process, the variable r is antecedent to A. Its value is found before that of A is known, and from it the area is computed. In this case r is called the **independent variable** and A is called the **dependent variable.**

The formula for the distance s which a body falls from rest in a time t is $s = 16\,t^2$. If the time it takes a ball to fall from rest to the bottom of a cliff is known, the distance which it falls can be found. Here t is the independent variable and s is the dependent variable.

If, however, we wish to find the radius of a circle whose area is known, then it is the variable A which is independent, and from it the corresponding dependent r is found. Similarly, by the use of the formula $s = 16\,t^2$, the time which it takes a body to fall any given distance may be computed, since $t = \frac{\sqrt{s}}{4}\cdot$ In this case s is the independent variable and t is dependent.

In general, when an expression is given involving two variables, say, x and y, one of these is more naturally looked upon as the one to which values are first assigned, and from which the values of the other are determined. That one is the independent variable.

In the equations $x = 2\,y^2 - 6\,y + 3$ and $x = 9\sqrt{y} + 4\,y$, y is the independent variable. In $y = 2\,x^4 - 6\,x^2 + 2$, x is independent.

It often occurs, however, that when the equation is not solved for either variable in terms of the other, there is no reason for considering one as dependent rather than the other. In that case we decide arbitrarily which one we will consider as independent. When we have decided which we shall so consider, it is often desirable to solve the equation for the dependent variable.

In the expression $x^2 + y^2 = 4$ we may equally well consider either x or y as independent. If x is selected as the independent variable, we solve for y, obtaining $y = \pm\sqrt{4 - x^2}$. This enables us to find the values of y from those of x more readily than we could from the first form.

EXERCISES

1. Express the volume of a sphere as a function of its radius. In this functional relation, which variable is regarded as independent?

2. Express the radius of a sphere as a function of its volume. In this functional relation, which variable is regarded as independent?

3. Using the relations of exercises 1 and 2, find (a) the volume of a sphere whose radius is 3 feet, (b) the radius of a sphere whose volume is $288\,\pi$ cubic feet.

4. A rectangle has one side 2 feet longer than the other. Express the functional relation between the area of the rectangle and the length of its shorter side. What is the area of the rectangle if its shorter side is 7 feet? In this relation which variable is regarded as independent?

5. Express the length of the shorter side of the rectangle in exercise 4 in terms of the area. Which variable is regarded as independent? Find the shorter side if the area of the rectangle is 4 square feet.

In finding pairs of values of x and y which satisfy the following equations, which variable in each case is naturally regarded as independent? Find three pairs of values of x and y satisfying each equation.

6. $x - 6\,y + 4\,y^2 + y^3 = 0.$

7. $x^4 - 8\,x^3 + 2\,y = 0.$

8. $y^2 + 2\,y - x^4 + 1 = 0.$

9. $xy + xy^2 + xy^3 = 12.$

10. $\sqrt{xy + 1} = y^3.$

11. $\dfrac{x^2 - y^2}{xy} - xy - xy^3 = 0.$

12. $x^{10} + 4 = y^4 - 4\,x^5.$

16. Accelerated motion. The distance s of a body from the ground t seconds after it has been thrown vertically upward with a velocity v_0 is a function of t:

$$s = v_0 t - 16\,t^2.*$$

To fix our ideas, suppose the velocity with which the body is thrown to be 64 feet per second. Then the equation becomes

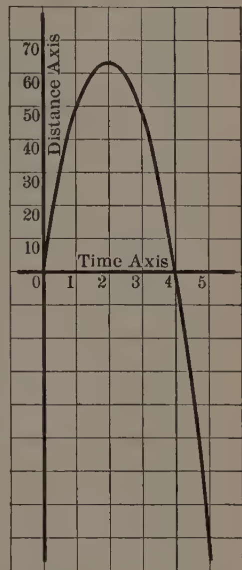
$$s = 64\,t - 16\,t^2. \tag{1}$$

If we wish to find how far above the ground the object is 3 seconds after the projection, we substitute 3 for t and compute the

* In this equation the resistance of the air is neglected. The number 16 is the approximate value of a constant depending on the force of gravity.

corresponding value of s. The relation between s and t may be shown by means of a graph, if the time axis and the distance axis are taken as in § 13. If we assign the values 0, 1, 2, 3, 4, 5 to t, and compute the value of s corresponding to each, we obtain the following table:

t	0	1	2	3	4	5
s	0	48	64	48	0	-80



We observe that when $t = 5$ the value of s is negative. This means that the object would be below the starting point at the end of 5 seconds if it were not stopped. It should be kept in mind that this graph does not picture the path of the body; it indicates that for the first 2 seconds its distance from the ground is increasing, and for the next 2 seconds it is decreasing, but the actual motion is in the same vertical line.

If we wish to know when the object is 60 feet from the ground, we may find from the graph the value of t when s equals 60. The result which we obtain by this means is only approximate on account of the inaccuracies of the drawing. If an exact value for t is desired, we can replace s by 60 in (1) and solve the resulting quadratic.

In graphing functions where the physical law represented is not in the foreground it is customary to use the variables x and y and to call the axes the X axis and the Y axis respectively.

EXERCISES

1. A stone is thrown vertically upward with an initial velocity of 48 feet per second. Plot its height as a function of the time. What is the height of the stone at the end of one second? two seconds? Determine from the graph how high it will rise, and the time required to reach the highest point. After how many seconds will it strike the ground?

2. Using one set of axes, plot the functional relation between the perimeter and the radius of a circle and also the relation between the area and the radius. Take the same unit distance along the vertical

axis to represent in the first a unit of length and in the second a square unit of area. What is indicated by the point of intersection of these graphs?

What do the graphs indicate as to the relative increase in the number of units of length and the number of units of area as the radius of the circle increases?

3. Answer the questions in exercise 2, using the surface and the volume of a sphere instead of the perimeter and the area of a circle.

4. The velocity of a body dropped from rest is given by the formula $v^2 = 64\,s$, where s is the distance fallen. Represent this law graphically. A ball is dropped from a height of 256 feet. How far must it fall to attain a velocity of 32 feet per second? With what velocity will it strike the ground?

5. If a body is projected downward with a velocity v_0, the distance s fallen at the end of any time t is given by the formula $s = v_0 t + 16\,t^2$. Express this law graphically when the velocity v_0 is 16 feet per second. How far will the body have fallen at the end of 2 seconds? How many seconds will it take to fall 60 feet? In how many seconds will the body reach the ground if projected from a height of 117 feet?

6. Work the preceding problem under the assumption that $v_0 = 0$, that is, that the body falls from rest.

17. Graphs of equations. The graph of an equation is the graphical representation of the functional relation which is expressed by the equation. For the purpose of graphing equations we agree:

I. To have at right angles to each other two lines: $X'OX$, called the ***X* axis**; and $Y'OY$, called the ***Y* axis.**

II. To call the point of intersection of the axes the **origin.**

III. To have a line of definite length for a unit of distance.

Then the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ to a distance $4\frac{1}{2}$ times the unit, etc.

IV. That the distance (measured parallel to the X axis) from the Y axis to any point in the surface of the paper be called the x distance, or **abscissa,** of the point, and that the distance (measured parallel to the Y axis) from the X axis to the point be called the y distance, or **ordinate,** of the point.

The values of the x distance and the y distance of a point are often called the **coördinates** of the point.

V. That the x distance of a point to the right of the Y axis be represented by a positive number, and the x distance of a point to the left by a negative number; also that the y distance of a point above the X axis be represented by a positive number, and the y distance of a point below the X axis by a negative number. Briefly, distances measured from the axes to the right or upward are positive, to the left or downward, negative.

VI. That every point in the surface of the paper correspond to a pair of numbers, one or both of which may be positive, negative, integral, or fractional.

VII. That of a given pair of numbers locating a point the first be the measure of the x distance and the second be the measure of the y distance.

Thus the point (2, 3) is the point whose x distance is 2 and whose y distance is 3.

Plotting or graphing an equation in x and y consists in finding the line or curve the coördinates of whose points satisfy the equation. The procedure is expressed in the following

RULE. *When y is alone on one side of the equation, set x equal to convenient integers and compute the corresponding values of y. Arrange the results in tabular form.*

Take corresponding values of x and y as coördinates and plot the various points.

Join the points in the order corresponding to increasing values of x, making the entire plot a smooth curve.

When x is alone on one side of the equation, integral values of y may be assumed and the corresponding values of x computed.

When the equation is not already solved for either x or y, either may be arbitrarily selected as the independent variable and the equation solved for the other. The resulting equation is plotted as already explained (see example 3 below). It should be noted that we obtain the same graph whichever variable we select as independent. The choice should be made so that the labor of solving for the dependent variable is as light as possible. For example, in case of the equation $x^2 - 3y^2 - 4y + 3 = 0$, we should take y for the independent variable. In $4x^2 - 7x - 2y = 0$ we should select x.

Care should be taken to join the points in the proper order so that the resulting curve pictures the variation of y when x increases through the values assumed for it.

Any convenient scale of units along the axes may be adopted. The scales should be so chosen that the portion of the curve which shows considerable curvature may be displayed in its relation to the axes and the origin.

When there is any question regarding the position of the curve between two integral values of x, an intermediate fractional value of x may be substituted, the corresponding value of y found, and thus an additional point obtained to fix the position of the curve in the vicinity in question.

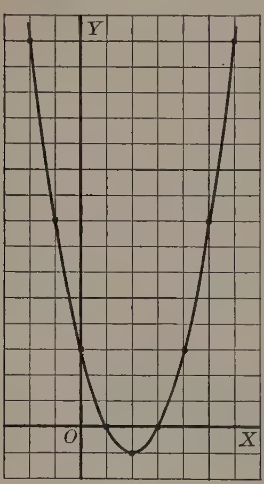
We shall assume without proof that the graph of a linear equation in two variables is a straight line. Hence in constructing the graph of such an equation we only need to locate two points whose coördinates satisfy the equation and then to draw a straight line through them. It is usually convenient to locate the two points where the line cuts the axes. If, however, these two points are close together, the direction of the line will not be accurately determined. Error can be avoided by selecting two points at a considerable distance apart.

The graphical solution of a system of two equations in two variables consists in plotting the equations to the same scale and on the same axes, and obtaining from the graph the values of x and y at each point of intersection.

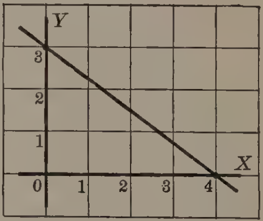
Two straight lines can intersect in but one point. Hence but one pair of values of x and y satisfies a system of two independent linear equations in two variables. When the two linear equations are incompatible their graphs are parallel lines.

EXAMPLES

1. Plot $3x + 4y = 12$.



Solution. If we set $x = 0$, we obtain $y = 3$. If we set $y = 0$, we obtain $x = 4$. That is, the points (0, 3) and (4, 0) lie on the line and serve to determine it completely.



2. Plot $x^2 - 4x + 3 = y$.

Solution. In this equation if we set $x = 0$, 1, 2, 3, etc., we obtain 3, 0, -1, 0, etc. as corresponding values of y. Thus the points (0, 3), (1, 0), (2, -1), (3, 0), etc. are on the curve. These points are joined in order by a smooth curve.

x	-2	-1	0	1	2	3	4	5	6
y	15	8	3	0	-1	0	3	8	15

3. Plot $2x^2 + 3y^2 = 9$.

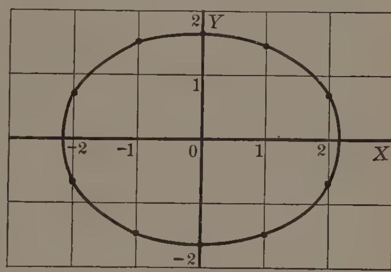
Solution.

$$3y^2 = 9 - 2x^2.$$
$$y^2 = 3 - \tfrac{2}{3}x^2.$$
$$y = \pm \tfrac{1}{3}\sqrt{27 - 6x^2}.$$

Assuming various integral values for x, we obtain the following table and plot; the values of y are given to the nearest tenth.

x	-3	-2	-1	0	1	2	3
y	imaginary	$\pm\frac{1}{3}\sqrt{3}=\pm.6$	$\pm\frac{1}{3}\sqrt{21}=\pm1.5$	$\pm\frac{1}{3}\sqrt{27}=\pm1.7$	$\pm\frac{1}{3}\sqrt{21}=\pm1.5$	$\pm\frac{1}{3}\sqrt{3}=\pm.6$	imaginary

In this example, when x is greater than 3 or less than -3, y is imaginary. Thus none of the curve is found outside a strip bounded by the lines $x = +3$ and $x = -3$.



To find exactly where the curve crosses the X axis, the equation may be solved for x, and the value of x corresponding to $y = 0$ found. Thus

$$x = \pm\sqrt{\tfrac{9}{2} - \tfrac{3}{2}y^2}.$$

If $y = 0$, $x = \pm\sqrt{\tfrac{9}{2}} = \pm 2.1$. These points are included in the graph.

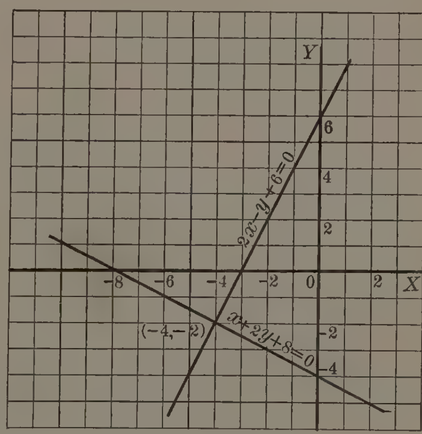
4. Solve graphically the system $\begin{cases} 2x - y + 6 = 0, \\ x + 2y + 8 = 0. \end{cases}$

Solution. Substituting 0 for x and then 0 for y in each equation, we obtain, for $2x - y + 6 = 0$,

x	0	-3
y	6	0

and for $x + 2y + 8 = 0$,

x	0	-8
y	-4	0



Then constructing the graph of each equation as indicated in the adjacent figure, we obtain, for the coördinates of the point of intersection of the two lines, $x = -4$ and $y = -2$.

EXERCISES

Plot the following equations:

1. $2x - 3y = 6$.
2. $3x + 7y + 14 = 0$.
3. $\frac{1}{2}y - \frac{1}{3}x = 2$.
4. $5.5x + 6y = 66$.
5. $4x - 3y = 0$.
6. $3x + 2y = 0$.
7. $y = x^2 - 3x - 4$.
8. $y = (x - 2)^2$.
9. $y = 5x - x^2$.
10. $y = 5x + x^2$.
11. $y = -5x - x^2$.
12. $y = -5x + x^2$.
13. $9x^2 + 4y^2 = 36$.
14. $9x^2 - 4y^2 = 36$.
15. $2x^2 + 2y^2 = 13$.
16. $xy = 4$.
17. $xy = -4$.
18. $x^2 + y^2 = 8$.
19. $x^2 - y^2 = 8$.
20. $y^2 - x^2 = 8$.
21. $x^2y^2 = 4$.

Plot the following systems and solve them graphically:

22. $\begin{cases} 3x - y = 4, \\ 2x + 4y = 19. \end{cases}$

23. $\begin{cases} 3x + y = 0, \\ 2y - x = 7. \end{cases}$

24. $\begin{cases} x^2 + y^2 = 25, \\ 3y - 4x = 0. \end{cases}$

25. $\begin{cases} y^2 - 4x = 17, \\ y - 2x = 1. \end{cases}$

26. $\begin{cases} xy = 3, \\ x - 4y = 4. \end{cases}$

CHAPTER III

QUADRATIC EQUATIONS

18. Solution by factoring. In order to solve most efficiently all kinds of quadratic equations, it is necessary to have two methods at command. The first method, that of factoring, is simpler to apply, and may be employed for the solution of many equations of higher degree. One should always observe whether an equation may be solved in this way before attempting the method of the next section.

The solution by factoring depends on the following

PRINCIPLE. *The product of two or more factors is zero when and only when one or more of the factors are zero.*

This principle is merely the formulation of the familiar rules for multiplication by zero. We know that if we multiply any number whatever by zero, the product is zero. If one factor of a product is zero, it makes no difference what numbers the other factors are; the product is zero. On the other hand, unless at least one of the factors of a product is zero, the product does not vanish.

It must be remembered that infinity is not a number and is never properly considered as such.

To illustrate the method of solution by factoring, consider the equation

$$x(x-3)(x-4)(x-1)=0. \tag{1}$$

We ask what must be the value of x in order that this equation may be satisfied; that is, what are the roots of this equation? Is 5 a root of the equation? It is not unless it satisfies the equation, and the equation is not satisfied unless at least one of its factors equals zero. But if we replace x by 5, the first factor becomes 5; the second, 2; the third, 1; and the fourth, 4; none of which is zero. Hence 5 is not a root of (1).

In seeking the roots of this equation we need only to consider the numbers which make one of the factors equal to zero. Hence x must be a number which will satisfy one of the four equations:

$$x = 0, \quad x - 4 = 0,$$
$$x - 3 = 0, \quad x - 1 = 0.$$

These are all linear equations whose roots are 0, 3, 4, 1 respectively. In accordance with the principle given above, these are the only roots of equation (1).

It is observed that by this method we have reduced the solution of equation (1), which is of the fourth degree, to the solution of a number of linear equations. To reduce the solution of a given equation to that of equations of lower degree is the essence of the method of solution by factoring. We may state the rule for solving an equation by factoring as follows:

RULE. *Transpose all the terms to the left member of the equation.*
Factor that member into linear factors.
Set each factor which involves the unknown equal to zero and solve the resulting equations.

19. Solution by formula. In order to obtain the formula which we shall use, it is necessary to solve the general quadratic equation

$$ax^2 + bx + c = 0, \qquad (Q)$$

where a, b, and c are real numbers, and where $a \neq 0$. This we do as follows:

Transposing c, $$ax^2 + bx = -c. \qquad (1)$$

Dividing by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Adding $\left(\frac{b}{2a}\right)^2$ to both members to make the left member a perfect square,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2}.$$

Expressing as a square, $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \qquad (2)$$

Extracting the square root,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}. \qquad (3)$$

Transposing, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (F)$$

The roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In equation (Q) the number x appears in a form which gives us no idea of its value in terms of a, b, and c. It is indeed unknown. But each step of the solution brings us nearer to an equation in which x stands alone in one member, which is the object of the process. The critical point in the procedure is passing from (2) to (3). Since the square root of any number or expression (not zero) has not one but two values, the necessity of extracting the square root in order to find the value of x carries with it the existence of two roots of the quadratic equation. As in the case of the linear equation, the process of solution does not change the value of x; it discovers it. The value of x in (Q) is knotted up in the equation, and we merely untangle the knots to display it in terms of known constants.

The solution of quadratic equations which involve fractions or radicals often necessitates the operation of multiplication by an expression involving x, or that of raising both sides of the equation to a power. Either operation may introduce into the equation roots which it did not originally possess, and lead to values of x which do not satisfy the original equation. Such results are called **extraneous** and should never be retained as roots. A certain method of detecting extraneous roots is to substitute in the original equation all the values of x which have been obtained, and retain only those which satisfy it.

To solve a quadratic equation in x by formula we proceed as follows:

RULE. *Write the equation in standard form* (Q).

Substitute the coefficient of x^2, *the coefficient of* x, *and the constant term for* a, b, *and* c, *respectively, in* (F).

If, in getting the equation into standard form, each member has been multiplied by an expression involving the unknown, or has been raised to a power, substitute in the original equation all the values which have been obtained, and reject the extraneous roots.

Even if neither of these operations has been employed, the substitution in the original equation of the values found should be performed in order to afford a check on the accuracy of the solution. A check which is more convenient for many cases will be derived in § 24.

In the following exercises the quadratic equations should always be solved by factoring when possible.

EXAMPLES

Solve and check:

1. $5x^2 + 4x = 12.$

Solution. Transposing, $5x^2 + 4x - 12 = 0.$

Factoring, $(5x - 6)(x + 2) = 0.$

Hence x must satisfy one of the equations

$$5x - 6 = 0, \quad x + 2 = 0.$$

$$x = \tfrac{6}{5}, \quad \text{or} \quad x = -2.$$

Check. $5(\tfrac{6}{5})^2 + 4(\tfrac{6}{5}) = 12, \quad 5(-2)^2 + 4(-2) = 12,$

$$\tfrac{36}{5} + \tfrac{24}{5} = 12, \qquad 20 - 8 = 12.$$

$$\tfrac{60}{5} = 12.$$

2. $x^2 + 2ab(a^2 + b^2) = (a + b)^2 x.$

Solution. Transposing, $x^2 - (a^2 + 2ab + b^2)x + 2ab(a^2 + b^2) = 0.$

Factoring, $[x - (a^2 + b^2)](x - 2ab) = 0.$

$$x = a^2 + b^2, \quad \text{or} \quad x = 2ab.$$

Check. $(a^2 + b^2)^2 + 2ab(a^2 + b^2) = (a + b)^2(a^2 + b^2).$

Dividing by $a^2 + b^2$, $a^2 + 2ab + b^2 = (a + b)^2.$

Also $(2ab)^2 + 2ab(a^2 + b^2) = (a + b)^2 \cdot 2ab.$

Dividing by $2ab$, $a^2 + 2ab + b^2 = (a + b)^2.$

3. $\sqrt{13 + x} + \sqrt{13 - x} = 6.$

Solution. Transposing, $\sqrt{13 + x} = 6 - \sqrt{13 - x}.$

Squaring, $13 + x = 36 - 12\sqrt{13 - x} + 13 - x.$

Transposing and dividing by 2, $x - 18 = -6\sqrt{13 - x}.$

Squaring, $$x^2 - 36x + 324 = 36(13 - x) = 468 - 36x.$$

$$x^2 = 144.$$

$$x = \pm 12.$$

$$\sqrt{13+12} + \sqrt{13-12} = 6, \quad \sqrt{13-12} + \sqrt{13+12} = 6,$$

$$5 + 1 = 6, \qquad 1 + 5 = 6.$$

Therefore 12 and -12 are roots.

4. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} = \dfrac{2\sqrt{x}}{\sqrt{a}+\sqrt{x}} - \dfrac{(x+a)^2}{a(x-a)}.$

Solution. Rationalizing the denominators,

$$\frac{a + 2\sqrt{ax} + x}{a - x} = \frac{2\sqrt{ax} - 2x}{a - x} - \frac{(x+a)^2}{a(x-a)}.$$

Now $a - x$ cannot equal 0, for this would give zeros in the denominators. Hence we can divide $a - x$ out of each denominator; then, multiplying through by a, we have

$$a^2 + 2a\sqrt{ax} + ax = 2a\sqrt{ax} - 2ax + x^2 + 2ax + a^2,$$

$$x^2 - ax = 0,$$

$$x(x - a) = 0,$$

from which $$x = 0, \quad \text{or} \quad x = a.$$

But $x = a$ has been excluded. Hence $x = 0$ is the only root.

Check. Substituting $x = 0$, we have $1 = 0 + 1.$

5. $3x^2 - 5x = 1.$

Solution. Writing in standard form,

$$3x^2 - 5x - 1 = 0.$$

Here 3 corresponds to a, -5 to b, and -1 to c in the general quadratic $ax^2 + bx + c = 0$. Substituting these values in (F),

gives $$x = \frac{-(-5) \pm \sqrt{25 - 4 \cdot 3(-1)}}{2 \cdot 3}$$

$$= \frac{5 \pm \sqrt{25 + 12}}{6} = \frac{5 \pm \sqrt{37}}{6}.$$

Check. $$3 \cdot \frac{25 \pm 10\sqrt{37} + 37}{36} - 5 \cdot \frac{5 \pm \sqrt{37}}{6} = 1,$$

$$31 \pm 5\sqrt{37} - 25 \mp 5\sqrt{37} = 6,$$

$$31 - 25 = 6.$$

6. $2k^2x^2 = kx + 2.$

Solution. Writing in standard form,

$$2k^2x^2 - kx - 2 = 0.$$

Then $a = 2k^2,\ b = -k,$ and $c = -2.$

Substituting these values in the formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-2)}}{2 \cdot 2k^2}$$

$$= \frac{k \pm \sqrt{k^2 + 16k^2}}{4k^2} = \frac{k \pm k\sqrt{17}}{4k^2} = \frac{1 \pm \sqrt{17}}{4k}.$$

Check. $$2k^2\left(\frac{1 \pm \sqrt{17}}{4k}\right)^2 = \frac{k(1 \pm \sqrt{17})}{4k} + 2,$$

$$\frac{18 \pm 2\sqrt{17}}{8} = \frac{1 \pm \sqrt{17} + 8}{4},$$

$$9 \pm \sqrt{17} = 9 \pm \sqrt{17}.$$

7. $\sqrt{x+1} + \sqrt{3x+1} - 2 = 0.$ (1)

Solution. Transposing, $\sqrt{3x+1} = 2 - \sqrt{x+1}.$ (2)

Squaring both members of (2), $3x + 1 = 4 - 4\sqrt{x+1} + x + 1.$ (3)

Transposing and collecting, $2x - 4 = -4\sqrt{x+1}.$ (4)

Dividing (4) by 2, $x - 2 = -2\sqrt{x+1}.$ (5)

Squaring both members of (5), $x^2 - 4x + 4 = 4x + 4.$

Transposing, $x^2 - 8x = 0.$

Factoring, $x(x - 8) = 0.$

Therefore $x = 0$ or $x = 8.$

Substituting 0 for x in (1), $1 + 1 - 2 = 0.$

Therefore 0 is a root of (1).

Substituting 8 for x in (1), $\sqrt{8+1} + \sqrt{24+1} - 2 = 0,$

$$3 + 5 - 2 = 0,$$

or $$6 = 0.$$

Therefore 8 is extraneous and 0 is the only root of (1).

NOTE. If both members of an equation are multiplied by an expression containing the variable, or are raised to a power, extraneous roots may be introduced. When either of these operations enters into a solution, the substitution of the results in the original equation is properly a part of the solution, as in examples 3 and 7 above. When these operations are not used, the substitution is merely a check on the accuracy of the work.

EXERCISES

Solve the following equations:

1. $x^2 - 4x - 21 = 0.$
2. $r^2 - 10r + 24 = 0.$
3. $t^2 + 10t - 24 = 0.$
4. $y^2 - 10y - 24 = 0.$
5. $z^2 + 10z + 24 = 0.$
6. $x^2 - 2x + 2 = 0.$
7. $6x^2 - 7x + 2 = 0.$
8. $s^2 - 10s + 18 = 0.$
9. $12x^2 - 71x - 6 = 0.$
10. $6x^2 + 5x = 56.$
11. $9y^2 = 6y + 26.$
12. $x^2 - .63x + .018 = 0.$
13. $3x^2 + 7 = 8x.$
14. $.03x^2 - 2.23x + 1.1075 = 0.$
15. $acx^2 - bcx + adx = bd.$
16. $4a + ax^2 = 2x + 2a^2x.$
17. $x^2 - 2ax + a^2 + b^2 = 0.$
18. $14x^2 + 45.5x = -36.26.$
19. $(x - a + b)(x - a + c) = (a - b)^2 - x^2.$
20. $a^2(b - x)^2 = b^2(a - x)^2.$
21. $(x - 6)^2 - (2x - 5)^2 = 6.$
22. $(2x - 17)(x - 5) - (3x + 1)(x - 7) = 84.$
23. $m^2x^2 - m(a - b)x - ab = 0.$
24. $x + a = (x^2 - x + 1)(x + a).$
25. $(2x - a)^2 = b(2x - a) + 2b^2.$
26. $(3x - 2a + b)^2 + 2b(3x - 2a + b) = a^2 - b^2.$
27. $x + \dfrac{1}{x} = a + \dfrac{1}{a}.$
28. $a + x = \dfrac{1}{a} + \dfrac{1}{x}.$
29. $cd(1 + x^2) = (c^2 + d^2)x.$
30. $k^2(lz^2 - 1) = l\left(2z^2 + z - \dfrac{k^2}{l}\right).$
31. $4mnx + (m^2 - n^2)(1 - x^2) = 0.$
32. $(x - 1)^2 = a(x^2 - 1).$
33. $(7 - 4\sqrt{3})x^2 + (2 - \sqrt{3})x = 2$
34. $\dfrac{2x - a}{b} + 3 = \dfrac{4a}{2x - b}.$
35. $\dfrac{1}{2 + \sqrt{4 - x}} + \dfrac{1}{2 - \sqrt{4 - x}} = \dfrac{2x}{9}.$

SUGGESTION. Rationalize the denominators.

36. $\dfrac{1}{1+\sqrt{1-x}}+\dfrac{1}{1-\sqrt{1-x}}=\dfrac{x}{2}.$

37. $\sqrt{11-x}+\sqrt{x-2}=3.$

38. $x\sqrt{x-2}+2\sqrt{x+2}=\sqrt{x^3+8}.$

39. $\sqrt{a(x-b)}+\sqrt{b(x-a)}=x.$

40. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

41. $\dfrac{4}{x-1}+\dfrac{1}{x-4}=\dfrac{3}{x-2}+\dfrac{2}{x-3}.$

42. $\dfrac{2}{4-x}-\dfrac{3}{5-x}=\dfrac{4}{6-x}-\dfrac{5}{7-x}.$

43. $\dfrac{\sqrt{a-x}}{x}-\dfrac{\sqrt{a-x}}{a}=\sqrt{x}.$

44. $2\sqrt{x^2-9x+18}-\sqrt{x^2-4x-12}=x-6.$

SUGGESTION. Factor the expressions under the radicals.

45. $\dfrac{x-7}{\sqrt{x-3}-2}+\dfrac{x-5}{\sqrt{x-4}-1}=4\sqrt{x-3}.$

SUGGESTION. Write the numerators $x-3-4$ and $x-4-1$.

46. $\sqrt{\dfrac{a+x}{b+x}}+\sqrt{\dfrac{a-x}{b-x}}=2\sqrt{\dfrac{a}{b}}.$

20. Quadratic form. An equation is in **quadratic form** if it may be considered as a trinomial consisting of a constant term and two terms involving the unknown (or an expression which may be considered as the unknown), the exponent of the unknown in one term being twice that in the other.

Thus $x-8\sqrt{x}+13=0$, $x^{-\frac{4}{3}}+x^{-\frac{2}{3}}-3=0$, $a^2x^{-2n}-(a+b)x^{-n}+b^2=0$, $x^2-2x-3-\sqrt{x^2-2x-3}+17=0$ are all in quadratic form. In the last the expression $\sqrt{x^2-2x-3}$ is taken as the unknown.

It is often convenient to replace by a single letter the lower power of the variable or expression with respect to which the equation is in quadratic form, and proceed as in the case of the ordinary quadratic equation.

EXAMPLES

1. Solve $x - 8\sqrt{x} + 15 = 0$.

Solution. This is a quadratic in $\sqrt{x}$ and may be written

$$(\sqrt{x})^2 - 8\sqrt{x} + 15 = 0.$$

Factoring, $$(\sqrt{x} - 5)(\sqrt{x} - 3) = 0.$$

$$\sqrt{x} = 5, \quad \sqrt{x} = 3.$$

The roots are $$x = 25, \quad x = 9.$$

Check. $25 - 8 \cdot 5 + 15 = 0$; $9 - 8 \cdot 3 + 15 = 0$.

2. Solve $(x^2 - x)^2 + x^2 - x - 6 = 0.$ (1)

Solution. This is a quadratic in $x^2 - x$.

Factoring, $$[(x^2 - x) - 2]\,[(x^2 - x) + 3] = 0,$$

or $$x^2 - x - 2 = 0, \tag{2}$$

$$x^2 - x + 3 = 0. \tag{3}$$

Factoring (2), $$(x + 1)(x - 2) = 0.$$

Hence $$x = -1, \quad x = 2.$$

Applying the formula to (3),

$$x = \frac{1 \pm \sqrt{1 - 4 \cdot 3}}{2} = \frac{1 \pm \sqrt{-11}}{2}.$$

The roots of (1) are $x = -1, 2, \dfrac{1 \pm \sqrt{-11}}{2}$.

Check. Substituting -1 in (1),

$$(1 + 1)^2 + (1 + 1) - 6 = 4 + 2 - 6 = 0.$$

Substituting 2 in (1),

$$(4 - 2)^2 + (4 - 2) - 6 = 4 + 2 - 6 = 0.$$

Substituting $\dfrac{1 \pm \sqrt{-11}}{2}$ in (1),

$$\left[\left(\frac{1 \pm \sqrt{-11}}{2}\right)^2 - \frac{1 \pm \sqrt{-11}}{2}\right]^2 + \left(\frac{1 \pm \sqrt{-11}}{2}\right)^2 - \frac{1 \pm \sqrt{-11}}{2} - 6 = 9 - 3 - 6 = 0.*$$

* The product $\sqrt{-a} \cdot \sqrt{-a} = \sqrt{a(-1)} \cdot \sqrt{a(-1)} = a(\sqrt{-1})^2 = -a$, where a is any positive number. The operations on complex numbers will be explained fully in Chapter V.

EXERCISES

Solve the following equations:

1. $2x + 7\sqrt{x} - 4 = 0.$
2. $x^{\frac{1}{4}} + 2x^{\frac{1}{2}} = 1.$
3. $x^{-1} + x^{-\frac{1}{2}} = \frac{3}{4}.$
4. $x^4 - 12x^2 + 27 = 0.$
5. $x^{\frac{1}{4}} - x^{-\frac{1}{4}} = 1\frac{1}{2}.$
6. $x^2 + x^{-2} = a^2 + a^{-2}.$
7. $x + 3\sqrt{5x} = 50.$
8. $7x^{\frac{1}{4}} - 3x^{\frac{1}{2}} - 2 = 0.$
9. $x - 1 - \sqrt{x+5} = 0.$
10. $x + \sqrt{x+3} = 4x - 1.$
11. $3x - 4\sqrt{x-7} = 2(x+2).$
12. $x + 1215 = 49\sqrt{615 + x}.$
13. $19x^{\frac{3}{2}} = x^3 - 216.$
14. $13x^{\frac{2}{3}} = x^{\frac{4}{3}} + 36.$
15. $x^2+3x-1-\sqrt{2x^2+6x+1}=0.$
16. $x^2 + 5x - 10 = \sqrt{x^2 + 5x + 2}.$
17. $\left(x + \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 10.$
18. $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4.$
19. $5x + \sqrt{5x - \frac{5}{x}} = \frac{5}{x}.$
20. $(2x^2-3x+1)^2=22x^2-33x+1.$
21. $\frac{\sqrt{x} - a}{b} = \frac{x}{(a+b)^2}.$
22. $\frac{\sqrt{x} + 2a - b}{a} = \frac{3a - b}{\sqrt{x}}.$
23. $\sqrt{x} + \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b}} = \frac{1}{\sqrt{x}} + \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a}}.$
24. $8(8x-5)^3 + 5(5-8x)^6 = 85.$
25. $(x^2+2)^{\frac{5}{2}} + \frac{3}{\sqrt{x^2+2}} = 4x^2 + 8.$

21. Number of roots. In § 19 the general quadratic equation $ax^2 + bx + c = 0$ was solved, and it was found that it has two roots. Reference to this solution shows that the roots found are the only ones possible, for none of the operations which we performed in the course of this solution affected the character of the result. For instance, if x satisfies (Q), it must also satisfy the equation $ax^2 + bx = -c$, for this is obtained from (Q) by adding $-c$ to each member. We may follow through all of the equations which we obtain in the course of the solution and see that any value of x which satisfies (Q) must satisfy each of them. But we finally obtain the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which tells us that x must have one of the two values which we obtain by taking separately the signs in the numerator before the

radical. Since this is the same equation which we had at first, except for its form, and since in this equation x can have only two values, the same is true for (Q), and the general quadratic has only two roots.

22. The factor theorem for the quadratic. Although this theorem will be proved later for equations of higher degree, a demonstration for the special case of the quadratic is included here.

THEOREM. *If x_1 is a root of the equation*

$$ax^2 + bx + c = 0, \tag{Q}$$

then $x - x_1$ is a factor of its left member.

The hypothesis in this theorem is that x_1 is a root of (Q); that is, the equation must be satisfied when x is replaced by x_1. Hence the hypothesis is equivalent to the statement that $ax_1^2 + bx_1 + c = 0$ (see § 11).

Consequently,

$$ax^2 + bx + c = ax^2 + bx + c - (ax_1^2 + bx_1 + c),$$

since $$ax_1^2 + bx_1 + c = 0.$$

But $ax^2 + bx + c - (ax_1^2 + bx_1 + c) = a(x^2 - x_1^2) + b(x - x_1)$.

Taking out the common factor, we obtain

$$(x - x_1)[a(x + x_1) + b].$$

Hence $x - x_1$ is a factor of the left member of (Q).

By means of this theorem we are able to write down a quadratic equation if its roots are known.

EXAMPLE

Form the equation whose roots are 4 and -2.

Solution. By the preceding theorem the factors of the left member of the equation must be $x - 4$ and $x + 2$.

Hence the equation is $(x - 4)(x + 2) = 0$, or $x^2 - 2x - 8 = 0$.

EXERCISES

Form the equation whose roots are the following:

1. $-3, 2$.
2. $\sqrt{3}, -\sqrt{3}$.
3. $2\sqrt{2}, -\sqrt{8}$.
4. $1, 2$.
5. $-1, -4$.
6. $0, 2$.
7. $1, 1$.
8. $2 + \sqrt{3}, 2 - \sqrt{3}$.
9. $1 + \sqrt{-1}, 1 - \sqrt{-1}$.
10. $\sqrt{-1}, -\sqrt{-1}$.
11. $\sqrt[4]{4}, -\sqrt[6]{8}$.
12. $\dfrac{3 \pm \sqrt{-31}}{4}$.

23. Reduced form of quadratic. If we multiply each member of an equation by a factor which involves the unknown, we obtain a new equation which has more roots than the original. The roots gained by this process are the values which reduce to zero the expression by which we multiply.

Multiplying $x^2 - 3x + 2 = 0$ by $x - 4$, we have $(x - 4)(x^2 - 3x + 2) = 0$, which is satisfied not only by the roots of the original equation but by the number 4 in addition.

Similarly, if we divide each member of an equation by a factor which involves the unknown, we obtain an equation with a less number of roots than the original. Here the roots which are lost are the values which reduce to zero the expression by which we divide.

For example, if $x^3 - 5x^2 + 6x = 0$ be divided by x, the equation loses the root $x = 0$.

We may, however, multiply or divide an equation by a constant, not zero, without affecting the number of its roots or the value of the unknown.

If we multiply both members of the equation $x^2 - 3x + 2 = 0$ by 4, we have $4(x^2 - 3x + 2) = 0$. This equation cannot be satisfied unless one of the factors of its left member becomes zero. The same values of x will make the expression inside the parenthesis vanish, regardless of any other factor which may be present, while the constant factor 4 is never zero. Hence the roots of the equation after multiplication by 4, or by any other constant, are the same as they were before.

We may accordingly divide the equation (Q) by the constant a without affecting the values of the roots of the equation.

The equation $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

may be written in the form $$x^2 + px + q = 0, \qquad (R)$$

where $$p = \frac{b}{a}, \quad \text{and} \quad q = \frac{c}{a}.$$

We shall call (R) the **reduced form** of (Q).

24. Relation between the roots and the coefficients. The equations (Q) and (R) have the same roots,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding these roots, we obtain

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a} = -p.$$

Multiplying the roots we get

$$x_1 \cdot x_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{b^2-b^2+4ac}{4a^2} = q.$$

These results we may state in the form of a

THEOREM. *The sum of the roots of a reduced quadratic equation equals the coefficient of the term in x, with its sign changed.*

The product of the roots of a reduced quadratic equation equals the constant term.

When roots of a quadratic equation are given in somewhat complicated form, it is simpler to form the equation by use of this theorem than by means of the theorem of § 22.

The above theorem also serves as a convenient check for the solution of a quadratic, especially when the roots are complicated.

EXAMPLES

1. Form the equation whose roots are $3+\sqrt{5}$ and $3-\sqrt{5}$.

Solution. Let $x_1 = 3+\sqrt{5}$, $x_2 = 3-\sqrt{5}$.

$$-p = x_1 + x_2 = 6, \text{ or } p = -6.$$

$$q = x_1 \cdot x_2 = 4.$$

The equation is $x^2 - 6x + 4 = 0$.

2. In the equation $x^2 - 2x + k = 0$,

what must be the value of k in order

(a) that one root shall be double the other?

(b) that the difference of the roots shall be half their sum?

Solution. (a) Let r and $2r$ represent the roots.

Then $$r + 2r = 3r = -p = 2, \quad (1)$$

and $$r \cdot 2r = 2r^2 = q = k. \quad (2)$$

From (1), $r = \frac{2}{3}$.

Substituting in (2), $k = \frac{8}{9}$.

Check. Putting $k = \frac{8}{9}$ in the original equation, we have

$$9x^2 - 18x + 8 = 0,$$

$$(3x-4)(3x-2) = 0,$$

$$x = \tfrac{4}{3}, \text{ or } x = \tfrac{2}{3},$$

and one root is double the other,

(b) Applying the formula, we have

$$x = \frac{2 \pm \sqrt{4 - 4k}}{2} = 1 \pm \sqrt{1 - k}.$$

The roots are $x_1 = 1 + \sqrt{1-k}$, $x_2 = 1 - \sqrt{1-k}$.

Since the difference of the roots is half their sum, and the sum of the roots is 2, we have

$$x_1 - x_2 = \frac{x_1 + x_2}{2} = \frac{2}{2},$$

or

$$1 + \sqrt{1-k} - (1 - \sqrt{1-k}) = 1.$$

$$2\sqrt{1-k} = 1,$$

$$1 - k = \tfrac{1}{4}, \quad k = \tfrac{3}{4}.$$

Check. Putting $k = \frac{3}{4}$ in the original equation, we have

$$4x^2 - 8x + 3 = 0,$$

$$(2x - 3)(2x - 1) = 0,$$

$$x = \tfrac{3}{2} \text{ or } \tfrac{1}{2}, \text{ and } \tfrac{3}{2} - \tfrac{1}{2} = \tfrac{1}{2}(\tfrac{3}{2} + \tfrac{1}{2}).$$

EXERCISES

1. Form the quadratic equation whose roots are $-2 + \sqrt{6}$ and $-2 - \sqrt{6}$.

2. Form the quadratic equation whose roots are $5 + 2\sqrt{-1}$ and $5 - 2\sqrt{-1}$.

3. Form the quadratic equation whose roots are $\dfrac{1 \pm \sqrt{7}}{3}$.

4. Form the quadratic equation whose roots are $\dfrac{3 \pm \sqrt{131}}{10}$.

5. Find the value of the literal coefficients in the following equations:

(a) $x^2 + bx - 9 = 0$. One root is 3.

Hint. Since 3 is a root, it must satisfy the equation.

(b) $x^2 + 4x + c = 0$. One root is 2.

(c) $ax^2 + 3x - 4 = 0$. One root is 2.

(d) $x^2 - bx - 6 = 0$. One root is -3.

(e) $2x^2 - 6x - c = 0$. One root is -4.

(f) $x^2 - 6x + c = 0$. One root is double the other.

(g) $x^2 + c = 0$. The difference between the roots is 8.

(h) $x^2 - 5x + c = 0$. One root exceeds the other by 3.

(i) $x^2 - 7x + c = 0$. The difference between the roots is 6.

(j) $x^2 - 6x + c = 0$. The difference between the roots is 4.

6. In the equation $ax^2 - 3x + k = 0$, what must be the value of k in order that the product of the roots shall be twice their sum?

7. Form the equation whose roots are the reciprocals of the roots of the equation $2x^2 - 5x + 3 = 0$.

8. If x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$, show that

$$\frac{1}{x_1} + \frac{1}{x_2} = -\frac{b}{c}.$$

9. Find the condition that one of the roots of the equation $x^2 + px + q = 0$ is double the other.

10. For what values of k is one of the roots of the equation $(k-4)x^2 - (2k-1)x = 7 - 5k$ double the other?

11. Find the condition that one of the roots of the equation $ax^2 + bx + c = 0$ is the reciprocal of the other.

HINT. Let x_1 be one root and $\frac{1}{x_1}$ be the other.

12. For what values of k and l will one of the roots of the equation $kx^2 + lx + k = 0$ be the reciprocal of the other?

13. For what values of k will the difference of the roots of the equation $5x^2 + 4x + k = 0$ equal the sum of the squares of the roots?

14. Find the equation whose roots are the reciprocals of the roots of the equation $ax^2 + bx + c = 0$.

15. Find the equation whose roots are double the roots of the equation $x^2 + px + q = 0$.

16. Find the equation whose roots are n times the roots of the equation $x^2 + px + q = 0$.

17. Find the equation whose roots are the negatives of the roots of the equation $x^2 + px + q = 0$.

18. Given the equation $x^2 - 3x + 5 = 0$. What is the equation whose roots are (a) the negatives of the roots of the given equation? (b) three times the roots of the given equation? (c) the reciprocals of the roots of the given equation?

19. Show that the condition that one root of $ax^2 + bx + c = 0$ shall be n times the other root is

$$b^2 = \frac{(n+1)^2}{n} \cdot ac.$$

20. From the result of the preceding exercise find the condition (a) that the two roots of the equation $ax^2 + bx + c = 0$ shall be equal; (b) that one shall be twice the other; (c) that one shall be three times the other. Write an equation illustrating each case.

21. Find the equation whose roots are each less by 2 than the roots of the equation $x^2 - 5x + 4 = 0$. Set the left member of each equation equal to y and plot.

22. Find the equation whose roots are each less by k than the roots of the equation $x^2 + px + q = 0$.

HINT. Let the roots of the given equation be x_1 and x_2. Then the roots of the required equation will be $x_1 - k$ and $x_2 - k$.

23. Find the equation whose roots are each greater by 1 than the roots of the equation $x^2 - 2x - 3 = 0$. Set the left member of each equation equal to y and plot.

24. Find the equation one of whose roots is less by k than the smaller root of the equation $x^2 + px + q = 0$, and the other of whose roots is greater by k than the larger root of the given equation.

25. Find the equation which has for one root a number 2 less than the smaller root of the equation $x^2 - x - 2 = 0$, and for the other root a number 2 greater than the larger root of the same equation. Set the left member of each equation equal to y and plot.

25. Classification of numbers. All the numbers of algebra are in one or the other of two classes, **real numbers** and **complex numbers.**

The real number will be left undefined, since an accurate definition involves questions too delicate to be considered here. Any number which can express the measure of a distance is real, as, for instance, $2, \frac{3}{4}, \sqrt{5}, \sqrt{3} - 7\sqrt{2}$, and π.

Real numbers are of two kinds, **rational** and **irrational.**

A rational number is a positive or a negative integer, or a number which may be expressed as the quotient of two such integers.

Any real number which is not rational is an irrational number.

A complex number is the indicated sum of a real number and a **pure imaginary,** where by a pure imaginary number we mean the indicated square root of a negative number.

26. Character of the roots of the quadratic. The determination of the character of the roots of a quadratic equation consists in finding to which of the foregoing classes of numbers the roots belong.

Consider the equation (Q) and its roots,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

The expression $b^2 - 4ac$ which appears under the radical sign is called the **discriminant** of the equation. An inspection of its value is sufficient to determine the character of the roots of the quadratic. No formal proof is necessary to see that the following statements are true.

I. *When $b^2 - 4ac$ is negative, the roots are complex numbers.*

II. *When $b^2 - 4ac = 0$, the roots are real and equal. In this case $x_1 = x_2 = -\dfrac{b}{2a}$.*

III. *When $b^2 - 4ac$ is positive, the roots are real and distinct.*

IV. *When $b^2 - 4ac$ is positive and a perfect square, the roots are real, distinct, and rational.*

The converses of these four cases are also true. For instance, (1) can only be complex when the expression under the radical sign is negative; that is, when the discriminant is negative.

27. Parameters. We often meet quadratic equations whose coefficients are not numerical, but involve one or more letters. For instance, the equation $x^2 + 2x + k = 0$ is of this type. Several of the equations on pages 50–51 are of the same kind. The letter k might conceivably have any value we choose to give it, but after we have once assigned a value, it is a fixed constant. Before we have decided what value to assign, it is indefinite. For each value of k there is a perfectly definite quadratic equation. If, for example, we give k the value -3, we have the corresponding equation $x^2 + 2x - 3 = 0$, whose roots we can find. But the equation $x^2 + 2x + k = 0$ really represents an infinite number of numerical equations corresponding to the infinite number of values which k may take on. Some of these may have complex roots and others may have real roots, and it is often necessary to select from this infinite set of equations the one, or the few, whose roots have a certain character. Symbols like this letter k are called **parameters**, to distinguish them from the letter x, which we have called the unknown or the variable. The whole set of equations which we obtain by letting the parameter take on a set of values, we call a family of equations.

EXAMPLE

For what values of k are the roots of $x^2 + k(x-1) + 3 = 0$ equal? real? complex?

Solution. Writing the equation in the form (Q), we have

$$x^2 + kx + 3 - k = 0,$$

where $$a = 1, \quad b = k, \quad \text{and} \quad c = 3 - k.$$

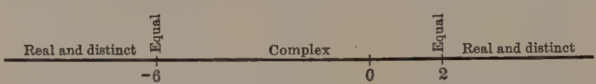
The discriminant $b^2 - 4ac = k^2 - 4(3-k) = k^2 + 4k - 12 = (k+6)(k-2)$.

The roots will be equal when $(k+6)(k-2) = 0$; that is, when $k = -6$ or 2.

The roots will be real when $(k+6)(k-2) > 0$. In this case the factors $k+6$ and $k-2$ must be both positive or both negative; that is, $k > 2$ or $k < -6$.

The roots will be complex when $(k+6)(k-2) < 0$. Hence one of the factors $k+6$, $k-2$, must be positive and the other negative. It appears that the factors are both positive when $k > 2$, and both negative when $k < -6$. But for values of k between these numbers the first factor is positive, while the second is negative. Hence the roots of the original equation are complex for values of k such that $-6 < k < 2$.

The situation may be illustrated by representing the values of k on a line as follows:



EXERCISES

1. Find, by use of the discriminant, the character of the roots of the following equations:

(a) $2x^2 - 7x + 3 = 0$.
(b) $9x^2 - 4x + \frac{4}{9} = 0$.
(c) $x^2 + 6x - 8 = 0$.
(d) $2x^2 - 4x + 3 = 0$.
(e) $18x^2 + 63x + 40 = 0$.
(f) $3t^2 + 15t + 19 = 0$.

2. For what values of k are the roots of $9x^2 + (1+k)x + 4 = 0$ equal? real? complex?

3. For what values of n are the roots of $8n^2x(x+3) = n - 5$ equal? real? complex?

4. For what values of k are the roots of $x^2 + k(x+1) + 3 = 0$ equal? real? complex?

5. What can be said of the character of the roots of the equation $ax^2 + bx + c = 0$ if a and c have opposite signs? Would changing the sign of b affect the character of the roots?

28. The special quadratics. In the general quadratic (Q) the letters b and c were supposed to have any real values. Since zero is included among these values, all our work has tacitly included the cases where one or more of the coefficients vanish, provided zero does not appear in the denominator of a fraction. If $a = 0$, the quadratic equation degenerates into a linear equation. This case will be considered in the next section.

Case I. We first consider the case where $c = 0$, and the equation reduces to the form
$$ax^2 + bx = 0.$$

Factoring, we obtain $\quad x(ax + b) = 0.$

Hence the roots are $\quad x = 0$, and $x = -\dfrac{b}{a}$.

That is, one of the roots is zero.

Conversely, if $x = 0$ is a root, $x - 0$ is a factor of the quadratic (see § 22), and the equation can have no constant term.

This result we may express by

Theorem I. *A quadratic equation has a root equal to zero when and only when its constant term is lacking.*

Case II. In case $b = 0$ we have the special equation
$$ax^2 + c = 0.$$

Writing the equation in the form
$$x^2 = -\frac{c}{a},$$
we obtain the roots $x_1 = \sqrt{-\dfrac{c}{a}}$, $x_2 = -\sqrt{-\dfrac{c}{a}}$, which are equal numerically but have opposite signs.

Conversely, when the roots of a quadratic equation are equal numerically but have opposite signs, the equation has no linear term in the unknown. For if we represent the roots by x_1 and $-x_1$, the corresponding factors are $x - x_1$ and $x + x_1$. The resulting equation is $x^2 - x_1^2 = 0$, which does not contain a linear term in x.

This affords

Theorem II. *The roots of the general quadratic equation are equal but with opposite signs when and only when $b = 0$.*

Case III. If $c = b = 0$, we have the special case $ax^2 = 0$, where the roots are equal by Case II, and hence each is zero by Case I.

EXERCISES

1. Prove Theorem I by considering the expressions for the roots in terms of the coefficients given in § 24.

2. (a) If the equation formed by setting a function of x equal to zero has a zero root, what is the characteristic property of the graph of the function?

(b) If the equation formed by setting a quadratic function of x equal to zero has roots equal but opposite in sign, what is the characteristic property of the graph of the function?

3. Determine k so that each of the following equations shall have one root equal to zero:

(a) $3x^2 - 2x + 2k^2 - 2 = 0.$
(b) $x^2 - x + 2k^2 + 3k - 2 = 0.$
(c) $5x^2 - 3x - k^2 - 12k - 5 = 0.$
(d) $(x - k)^2 + 3(x - 2k) = 0.$
(e) $(3x + k - 1)^2 - 2(3x + k - 1) + 1 = 0.$

4. Determine k and m so that each of the following equations shall have both roots equal to zero:

(a) $2x^2 + 3kx + 7mx - x + k + m + 1 = 0.$
(b) $x^2 + 3kx + 4mx = m - 5k + 253x.$
(c) $3x^2 + k(x - 2) + mx + 1 + k^2 = 0.$
(d) $m(x^2 - x + 1) + kx = x - m^2 + 2.$
(e) $m^3x(1 + x) - (1 + 3m^2)x - m(2 - 3x) + k = 0.$

5. Determine k so that each of the following equations shall have roots numerically equal but opposite in sign:

(a) $x^2 - 2k^2x = kx + 1.$
(b) $(9x + 5k^2)(x + k) = 2x.$
(c) $2k^2(x^2 + x + 1) - 5(kx + 3) + 2x = 0.$
(d) $x^2 + k^2(k - 1)x - 6(kx + 1) = 0.$
(e) $(x + 1)(kx - 1) = (1 - x)(1 + kx).$

29. Degeneration of the quadratic equation. If in the equation (Q) we set $a = 0$, while b and c are not zero, we no longer have a quadratic, but a linear equation, which has but one root. If instead of substituting 0 for a we let a take on smaller and smaller values, we shall obtain a number of equations of the same family whose

left members differ from the left members of the linear equation $bx + c = 0$ by just as little as we please. In this way we can find out what becomes of one of the roots of the quadratic when a vanishes. The roots of (Q) are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Since division by 0 is ruled out of algebra, we cannot replace a by 0 in these formulas. We can only let a approach 0. But even then x_1 approaches the form $\frac{0}{0}$, which is also meaningless. To avoid this difficulty we rationalize the numerators of both x_1 and x_2 by multiplying both numerator and denominator by a rationalizing factor of the numerator. We obtain

$$x_1 = \frac{(-b + \sqrt{b^2 - 4ac})}{2a} \cdot \frac{(-b - \sqrt{b^2 - 4ac})}{(-b - \sqrt{b^2 - 4ac})}$$

$$= \frac{b^2 - b^2 + 4ac}{2a(-b - \sqrt{b^2 - 4ac})} = \frac{2c}{-b - \sqrt{b^2 - 4ac}}.$$

$$x_2 = \frac{(-b - \sqrt{b^2 - 4ac})}{2a} \cdot \frac{(-b + \sqrt{b^2 - 4ac})}{(-b + \sqrt{b^2 - 4ac})}$$

$$= \frac{b^2 - b^2 + 4ac}{2a(-b + \sqrt{b^2 - 4ac})} = \frac{2c}{-b + \sqrt{b^2 - 4ac}}.$$

As a approaches 0, $b^2 - 4ac$ approaches b^2, and x_1 approaches $\frac{2c}{-b - \sqrt{b^2}}$ or $-\frac{c}{b}$. But since the denominator of x_2 becomes very small as a approaches zero, x_2 increases without limit, that is, becomes infinite. Thus in the equation $ax^2 + bx + c = 0$, when a is allowed to approach 0, one of the roots of the quadratic approaches the root of the linear equation $bx + c = 0$, while the other becomes infinite. The graph must then approach a straight line as a limit as a approaches 0. This is made clear from the following figure, which represents the equations of the family

$$ax^2 - \frac{x}{2} - 2 = y, \tag{1}$$

corresponding to the values $a = 1, \frac{1}{5}, \frac{1}{10}, \frac{1}{50}, 0, -\frac{1}{32}$.

In the figure the curves represent the following equations:

$$x^2 - \frac{x}{2} - 2 = y. \tag{I}$$

$$\frac{x^2}{5} - \frac{x}{2} - 2 = y. \tag{II}$$

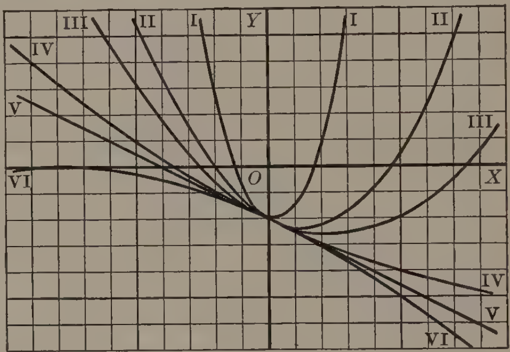
$$\frac{x^2}{10} - \frac{x}{2} - 2 = y. \tag{III}$$

$$\frac{x^2}{50} - \frac{x}{2} - 2 = y. \tag{IV}$$

$$-\frac{x}{2} - 2 = y. \tag{V}$$

$$-\frac{x^2}{32} - \frac{x}{2} - 2 = y. \tag{VI}$$

In a similar manner we can show that when in the equation $bx + c = 0$, b approaches 0 as a limit, the root of the linear equation becomes infinite (see II, § 28).



30. Families of curves. Equation (1) of the preceding section represents a family of equations, and the graph shows six of the corresponding family of curves. If we may judge by the four curves (I)–(IV), which appear in the figure, all the curves of the family are tangent to the same line. We might wish to know whether any equations of the family have equal roots. This algebraic question corresponds to the geometric question whether any of the curves of the family are tangent to the X axis. Apparently none of the curves tangent to the line (V) on its upper side has this property, but if we use the method of § 26, we find that if $a = -\frac{1}{32}$, equation (1) has equal roots. The graph of the equation $-\dfrac{x^2}{32} - \dfrac{x}{2} - 2 = y$ is denoted by (VI) in the figure. Since the constant term is not 0, no member of this family has a root equal to 0; that is, none of the curves passes through the origin.

EXERCISES

1. Show that all the curves of the family $y = ax^2 - \frac{x}{2} - 2$, considered in the preceding section, are tangent to the straight line $y = -\frac{x}{2} - 2$ at the point $(0, -2)$.

2. Plot several curves of the family $y = ax^2 + x + 1$. Discuss the behavior of these curves as a approaches zero. Will any of the curves go through the origin? For what value of a will the equation $ax^2 + x + 1 = 0$ have equal roots? What is the common tangent line of the family? What are the coördinates of the point of tangency?

3. Draw several curves of the family $y = ax^2 - 4$ and discuss the behavior of these curves as a varies.

4. Draw several curves of the family $y = ax^2 - 2x + 2$ and discuss the behavior of these curves as a varies.

5. What value must k approach so that one root of each of the following equations may become infinite?

(a) $2kx^2 - 6x^2 + 7x - k = 0$.
(b) $(x+1)(kx+x-1) = 1$.
(c) $(k+2)(x^2+1) = 2x(x-1)$.
(d) $k^2x^2 = 3(2k-3)(x^2-x-1)$.

31. Graphical solution of the quadratic equation. Let

$$y = ax^2 + bx + c, \tag{1}$$

where, as usual, a, b, and c represent real numbers and a is positive.

If we let x take on various values, y will have corresponding values and we may plot the equation as in § 17. A root of the quadratic equation

$$ax^2 + bx + c = 0 \tag{2}$$

is a number which substituted for x satisfies the equation and therefore gives the value $y = 0$ in (1). Thus the points on the graph of (1) which represent the real roots of equation (2) are the points for which $y = 0$; that is, where the curve crosses the X axis. The numerical values of these roots are the measures of the distances along the X axis from the origin to the points where the curve cuts the axis. The existence of complex roots of (1) is determined by the following

THEOREM. *If the graph of* (1) *has no point in common with the X axis, equation* (2) *has complex roots, and conversely.*

Every equation of form (2) has two roots either real or complex (§ 21). If the graph of (1) has no point in common with the X axis,

there is no *real* value of x for which $y=0$, and consequently no real root of (2). The roots must then be complex.

Conversely, if (2) has complex roots, there is no real value of x which satisfies it, and which makes $y=0$ in (1). Thus the curve has no point in common with the X axis.

32. Maxima and minima. Consider the equation

$$y = 2x^2 + 7x + 2. \tag{1}$$

By substituting for x a very large positive or negative number, say, $x = \pm 100$, y is large positively. Thus for values of x far to the right or left the curve lies far above the X axis, but for one value of x we get only one value of y; that is, there is only one point on the curve above (or below) any specified point on the X axis.

If, however, we assign to y a certain value, we can find the corresponding values of x by the solution of a quadratic equation; that is, the curve has two points, whose abscissas are either real, coincident, or complex, on the same horizontal line with any point on the Y axis. In equation (1) let $y=2$.

Then
$$2 = 2x^2 + 7x + 2,$$
or
$$2x^2 + 7x = 0.$$

The roots are $x_1 = -3\frac{1}{2}$, $x_2 = 0$.

Hence the points $(-3\frac{1}{2}, 2)$ and $(0, 2)$ are on the curve (§ 17); that is, if we go up two units on the Y axis, the curve is to be found $3\frac{1}{2}$ units to the left and also again on the Y axis. If in (1) we let $y=-4$, the corresponding values of x, namely, $-1\frac{1}{2}$ and -2, are very nearly equal to each other, which means that the curve meets a line parallel to the X axis and four units below it at points near together.

We may now ask, Where is the lowest, or minimum, point of the curve? This lowest point certainly has as its value of y that number to which correspond equal values of x. Hence we must determine for what value of y the equation (1), which we now write in the form

$$2x^2 + 7x + (2 - y) = 0,$$

has equal roots. Comparing this equation with $ax^2 + bx + c = 0$, we have
$$2 = a, \quad 7 = b, \quad 2 - y = c.$$

Thus the condition $b^2 - 4ac = 0$ becomes

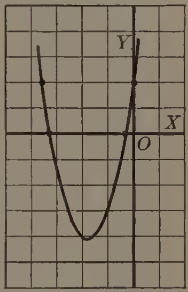
$$49 - 4 \cdot 2(2 - y) = 0,$$

or
$$y = -\frac{49 - 16}{8} = -\frac{33}{8} = -4\tfrac{1}{8}.$$

Substituting this value of y in (1), we get $-\frac{7}{4}$ as the corresponding value of x. We may express the foregoing results in tabular form, and draw the curve.

y	x
0	$-.3+$ or $-3.2+$
2	0 or $-3\frac{1}{2}$
-4	$-1\frac{1}{2}$ or -2
$-4\frac{1}{8}$	$-1\frac{3}{4}$

This gives a single value of y for which the values of x are equal; hence the graph of (1) is a single festoon, as in the figure.



If we take the equation

$$ax^2 + bx + c = y,$$

where a is positive, we find in a similar manner that the coördinates of the minimum point of the curve are

$$y = -\frac{b^2 - 4ac}{4a}, \tag{2}$$

$$x = -\frac{b}{2a}. \tag{3}$$

The results of this section enable us to determine from the coefficients the value of y for the lowest point of the curve, and hence to show beyond question whether the equation has real or complex roots. The ordinate of the lowest point is the least value that the function $ax^2 + bx + c$ takes on for any value of the variable.

When a is negative, the form of the graph is the same as that illustrated in the figure, but the curve is inverted, as in the case of curve (VI), p. 58. When a is negative we find the highest or maximum point of the curve in the same way that we have just found the minimum point for the case where the coefficient of x^2 is positive. The coördinates of the maximum point of the curve are the same as given in (2) and (3).

When $x = -\frac{b}{2a}$, the function $ax^2 + bx + c$ takes on its minimum value if a is positive, and its maximum value if a is negative. These maximum and minimum values of the function are the values of y given by the formula

$$y = -\frac{b^2 - 4ac}{4a}.$$

EXERCISES

Plot the following equations and determine the points where the graphs cut the X axis. Find in each case the lowest or highest point according as the curve is concave upward or downward.

1. $y = x^2 - 6x + 5.$
2. $y = x^2 + 4x + 4.$
3. $y = x^2 - 6x + 10.$
4. $y = 2x^2 - x - 3.$
5. $y = 1 - x - 2x^2.$
6. $x^2 + 4x - 2y = 0.$
7. $y = 2x - x^2.$
8. $y + x^2 + 2x + 2 = 0.$
9. $x^2 - 4x + 4 + 4y = 0.$
10. $x^2 - 5x - y = 0.$

11. Divide 10 into two parts such that their product shall be a maximum.

12. Divide 10 into two parts such that the sum of their squares shall be a minimum.

13. Divide 12 into two parts such that the product of half one part by a third of the other part shall be a maximum.

14. Find the number of acres in the largest rectangular field that can be inclosed by a mile of fence.

15. A window is to be made in the shape of a rectangle surmounted by an equilateral triangle one of whose sides is the upper base of the rectangle. The perimeter of the window is to be 22 feet. Find its width and height in order that it may admit the maximum amount of light.

Solution. Let the base of the rectangle be $2x$ and its altitude be y. Then the perimeter of the entire window is

$$2y + 6x = 22,$$

from which

$$y = 11 - 3x.$$

The area of the window is $2xy + \sqrt{3}x^2$,

or, substituting the value of y found above,

$$22x - 6x^2 + \sqrt{3}x^2.$$

The question is, For what value of x will this function take on its maximum value?

The coefficient of x^2 in this quadratic function is $\sqrt{3} - 6$ and the coefficient of x is 22. Substituting these values for a and b respectively in formula (3), we have

$$x = -\frac{b}{2a} = -\frac{22}{2(\sqrt{3}-6)} = -\frac{11}{\sqrt{3}-6} \cdot \frac{\sqrt{3}+6}{\sqrt{3}+6} = -\frac{11(\sqrt{3}+6)}{-33}$$

$$= \tfrac{1}{3}(\sqrt{3}+6) = \tfrac{1}{3}(1.7321 + 6) = 2.5774.$$

$2x = 5.15$ feet, the width of the window.

The height of the window is

$$\begin{aligned} y+\sqrt{3}x &= 11-3x+\sqrt{3}x \\ &= 11-(3-\sqrt{3})x = 11-(1.2679)(2.5774) \\ &= 11-3.27 = 7.73 \text{ feet.} \end{aligned}$$

16. Solve the same problem for a window in the shape of a rectangle surmounted by a right triangle whose hypotenuse is the upper base of the rectangle, the perimeter of the window being 28 feet.

17. Solve the same problem for a window in the shape of a rectangle surmounted by a trapezoid each of whose legs and upper base are equal to half the upper base of the rectangle, the perimeter of the window being 37.3 feet.

18. Solve the same problem for a window in the shape of a rectangle surmounted by a semicircle, the perimeter of the window being 32 feet.

19. Find the dimensions of the rectangle of largest area that can be inscribed in an isosceles triangle whose altitude is 20 and whose base is 14, one side of the rectangle lying on the base of the triangle.

20. Find the dimensions of the rectangle of largest area that can be inscribed in a right triangle whose legs are a and b, one angle of the rectangle coinciding with the right angle of the triangle.

CHAPTER IV

INEQUALITIES

33. General theorem. We say that a is greater than b, or $a > b$, when $a - b$ is positive. If $a - b$ is negative, then a is less than b, or $a < b$. As we distinguished between identities and equations of condition in § 11, so in this discussion we observe that some statements of inequality are true for any real value of the letters, while others hold for particular values only. The former class may be called **unconditional,** the latter **conditional,** inequalities.

Thus $a^2 > -1$ is true for any real value of a and is unconditional, while $x - 1 > 2$ only when x is greater than 3 and is consequently conditional.

The two inequalities $a > b$, $c > d$, are said to have the same **sense.** Similarly, $a < b$, $c < d$, have the same sense. The inequalities $a > b$, $c < d$, have different senses.

THEOREM. *Any **real** number may be added to or subtracted from each member of an inequality without affecting its sense. Both members of an inequality may be multiplied or divided by any **positive** number without affecting the sense of the inequality.*

Let $a > b$, that is, let $a - b = k$, where k is a positive number.

If m is any real number, evidently

$$a \pm m - (b \pm m) = k,$$

or,
$$a \pm m > b \pm m.$$

Similarly,
$$ma - mb = mk,$$

or, if m is positive,
$$ma > mb.$$

When m is less than 1 this amounts to dividing both members by a positive constant.

If each member of an inequality is multiplied by a negative number, that is, if m is negative, the sense of the inequality is changed.

COROLLARY I. *Terms may be transposed from one member of an inequality to the other, as in the case of equations.*

Corollary II. *If both members of an inequality are positive, each member may be raised to any power without changing the sense of the inequality; if both members are negative and each is raised to the same even power, the sense of the inequality is changed; if both members are negative and each is raised to the same odd power, the sense of the inequality is not changed.*

Thus, when both a and b are positive, if $a < b$, then $a^n < b^n$, but since $-a > -b$, $(-a)^3 > (-b)^3$ and $(-a)^2 < (-b)^2$.

Similarly, $\sqrt{a} < \sqrt{b}$, but $-\sqrt{a} > -\sqrt{b}$. That is, if the negative signs are taken in extracting the square root, the sense of the inequality is changed.

EXAMPLES

1. Show that $\dfrac{x^2 + 3y^2}{2y} > x + y$, unless $x = y$, where x and y represent positive real numbers.

Solution. Multiply both sides of the inequality by $2y$. This will not change the sense of the inequality since $2y$ is positive.

$$x^2 + 3y^2 > 2xy + 2y^2.$$

Subtracting $2xy + 2y^2$ from both sides,

$$x^2 - 2xy + y^2 > 0,$$

or

$$(x - y)^2 > 0.$$

This last inequality is true unless $x = y$, since the square of any real number except 0 is positive.

If now the steps are performed in the reverse order, the original inequality is established, and therefore holds for all positive real values of x and y unless $x = y$.*

2. Show that $a^3 + b^3 > a^2b + ab^2$, unless $a = b$, where a and b represent positive real numbers.

First solution. Divide each side of the inequality by $a + b$. Since $a + b$ is positive, the sense of the inequality is not changed.

$$a^2 - ab + b^2 > ab.$$

Subtracting ab from each side, $(a - b)^2 > 0$,

which is true unless $a = b$. Hence, reversing the order of the operations, it appears that the first inequality holds.

* In this method of proof we first assume that the inequality in question is true and then pass from it, by legitimate operations, to a self-evident inequality. But this process does not establish the validity of the original expression. The proof is not complete, until, starting with the evident inequality, we perform the operations which will lead back to the original. It is usually sufficient to observe that it is possible to go through this retrograde process without actually doing it.

Second solution. Subtract $a^2b + ab^2$ from each side.

$$a^3 - a^2b - ab^2 + b^3 > 0.$$

Factoring,
$$a^2(a-b) - b^2(a-b) > 0,$$
$$(a^2 - b^2)(a-b) > 0.$$

If $a > b$, both factors are positive. If $a < b$, both factors are negative. In either case their product is positive. Hence the inequality holds unless $a = b$.

3. Show that $\sqrt{3} + \sqrt{15} < \sqrt{10} + \sqrt{6}$. (1)

Solution. Squaring, $18 + 2\sqrt{45} < 16 + 2\sqrt{60}$. (2)

Transposing and dividing by 2, $1 + \sqrt{45} < \sqrt{60}$. (3)

Squaring, $46 + 2\sqrt{45} < 60$, (4)

$$2\sqrt{45} < 14,$$
$$\sqrt{45} < 7,$$

which is a known relation.

Now performing these operations in the reverse order, and taking the positive square root in passing from (4) to (3), and from (2) to (1), we find that the first inequality holds.

EXERCISES

Show that the following inequalities hold where the letters represent positive real numbers:

1. $\dfrac{2xy}{x+y} < \dfrac{x+y}{2}$, unless $x = y$.

2. $\left(\dfrac{a}{b} + 1\right)^2 > 4\dfrac{a}{b}$, unless $a = b$.

3. $1 - x - x^2 + x^3 > -4x - 4x^2$, unless $x = 1$.

4. $x + y < \dfrac{x^2}{y} + \dfrac{y^2}{x}$, unless $x = y$.

5. $a^2 + b^2 + c^2 > ab + bc + ca$, unless $a = b = c$.

6. $(a+b)(b+c)(c+a) > 8abc$, unless $a = b = c$.

7. $a^5 + b^5 > a^4b + ab^4$, unless $a = b$.

8. $x^3 + 1 > x^2 + x$, unless $x = 1$.

9. $x^{2n} + 1 > x^{2n-1} + x$, unless $x = 1$.

10. $\sqrt{7} + \sqrt{11} > \sqrt{5} + \sqrt{13}$.

11. $\sqrt{7} + 2\sqrt{3} < \sqrt{6} + \sqrt{14}$.

12. $\sqrt{3} + \sqrt{21} < 2\sqrt{10}$.

13. $\sqrt{2} + \sqrt{3} + \sqrt{5} < \sqrt{30}$.

14. $\sqrt{7} + \sqrt{11} < \sqrt{5} + \sqrt{14}$.

15. If $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$, prove that $ax + by < 1$.

16. Show that the sum of any positive number and its reciprocal is never less than 2.

17. Show that $x+1<2x^3$ if $x>1$, and that $x+1>2x^3$ if $x<1$.

34. Conditional linear inequalities. We have solved the equation $ax+b=0$, and found that $x=-\frac{b}{a}\cdot$ But if we consider the left member as a function of x, we see that for various values of x the expression $ax+b$ takes on different values, some of which may be greater and others less than 0. We now seek to determine the class of numbers which make

$$ax+b<0. \tag{1}$$

That is, we wish to solve this inequality.

First, let a be positive.

By Theorem, § 33, we have $ax<-b$,

hence $$x<-\frac{b}{a}.$$

Now let a be negative and equal to $-A$, where A is positive.

Then (1) may be written

$$-Ax+b<0, \quad \text{or} \quad Ax-b>0.$$

Solving as before, we obtain

$$x>\frac{b}{A}=\frac{b}{-a}=-\frac{b}{a}.$$

We may solve in a similar manner the inequality

$$ax+b>0. \tag{2}$$

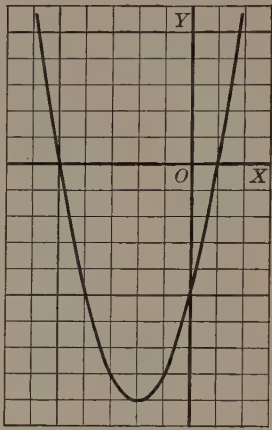
35. Graphical interpretation of the linear inequality. If we set $ax+b=y$, we see that if y is 0 the corresponding value of x must be the root of the equation $ax+b=0$. But all the values of x which give y a negative value satisfy the inequality (1); that is, the values of x for which the line $ax+b=y$ is below the X axis satisfy (1), while the values of x for which the line is above the X axis satisfy the inequality $ax+b>0$. Hence to solve an inequality of type (1) or (2) graphically we may plot the function represented by the left member, and determine for what values of x the graph is respectively below or above the X axis.

36. Conditional quadratic inequalities. Consider the expression

$$x^2+4x-5=y.$$

Construct the graph of this equation. From the figure it appears that y is negative for values of x between the roots of $x^2+4x-5=0$,

and positive for other values of x except the roots themselves. Since the roots are -5 and $+1$, we can say that the inequality $x^2+4x-5<0$ is satisfied when $-5<x<1$.



This example shows that if the equation $ax^2+bx+c=0$ has real roots, so that the corresponding graph cuts the X axis, and if a is positive, so that the curve is concave upwards, the inequality

$$ax^2+bx+c<0 \qquad (1)$$

is satisfied for values of x which lie between the values of the roots.

When the roots of $ax^2+bx+c=0$ are complex, so that the graph lies entirely above the X axis, there is no real value of x which satisfies (1).

When the roots are real and the sign of a is negative, so that the graph is concave downwards, the curve will be above the X axis for values of x between the roots, and hence (1) will be satisfied only by values of x exceeding the greater or less than the smaller root.

EXERCISES

Solve the following inequalities. Illustrate graphically:

1. $2x-5>0$.

2. $3x<5x+2$.

3. $3(x+1)>x+3$.

4. $3-4x>2-x$.

5. $(x+1)(x+6)<(x+2)(x+4)$.

6. $x^2<9$.

7. $1+4x+4x^2>0$.

8. $1+x+2x^2>0$.

9. $x+\frac{1}{x}<2$.

10. $(x-1)(x-4)<0$.

11. $x^2-x<6$.

12. $x^2+4x+3<0$.

13. $(1-x)(x-4)>0$.

14. $6+x>x^2$.

15. $x^2-8x+22>6$.

16. $x^2+3>3x$.

For what values of x do the following pairs of inequalities hold? Illustrate graphically:

17. $\begin{cases}3x+7<9+x,\\5x>9-4x.\end{cases}$

18. $\begin{cases}2(1-x)>3x+7,\\3x+2<5x+8.\end{cases}$

19. $\begin{cases}x^2+5<6x.\\3x-6>4x-9.\end{cases}$

20. $\begin{cases}x^2+2x<8,\\2x+8>x^2.\end{cases}$

CHAPTER V

COMPLEX NUMBERS

37. The imaginary unit. When we approached the solution of quadratic equations we saw that many equations, as, for example, the equation $x^2 = 2$, are not solvable if we are at liberty to use only rational numbers. It is necessary to introduce irrational numbers in order to solve them. The excuse for introducing such numbers is not that we need them as a means for more accurate measurement, — the rational numbers are entirely adequate for all mechanical purposes, — but that they are a mathematical necessity if we propose to solve equations of this type.

A similar situation demands the introduction of still other numbers. In attempting to solve the equation

$$x^2 = -1 \tag{1}$$

we may write the result in the form

$$x = \pm\sqrt{-1}.$$

But we realize that there is no real number whose square is -1. We may write the *symbol* $\sqrt{-1}$, but its meaning must be somewhat remote from that of $\sqrt{2}$, for in the latter case we have a process by which we can extract the square root and get a number whose square is as nearly equal to 2 as we desire. This process is not applicable in the case of $\sqrt{-1}$. In fact, this symbol differs from 1 or any real number not merely in degree but in kind. One cannot say that $\sqrt{-1}$ is greater or less than a real number, any more than one can compare the magnitudes of a quart and an inch.

$\sqrt{-1}$ is symbolized by i and is called the **imaginary unit.** The term "imaginary" is perhaps too firmly established in mathematical literature to warrant its discontinuance. It should be kept in mind, however, that it is really no more and no less imaginary than the negative or the irrational numbers. So far as we have yet proceeded it is merely something which satisfies equation (1). But when we have defined the various operations on it and ascribed to it the

characteristic properties of numbers, we shall be justified in calling it a number.

Just as we built up from the unit 1 a system of real numbers, so we shall construct from $\sqrt{-1} = i$ a system of imaginary numbers. The fact that we cannot measure $\sqrt{-1}$ on a rule will cause no more confusion than our inability to measure $\sqrt[3]{2}$ exactly. As we are able to deal with irrational numbers as readily as with integers when we define what we mean by the four rational operations on them, so will the imaginaries become indeed numbers with which we can work when we have defined for them the corresponding operations.

38. Addition and subtraction of imaginary numbers. We write

$$\begin{aligned} 0 &= 0\,i, \\ i &= 1\,i, \\ i + i &= 2\,i, \\ i + i + i &= 3\,i, \\ &\cdots\cdots\cdots \\ \underbrace{i + i + \cdots + i}_{n \text{ terms}} &= ni. \end{aligned} \tag{I}$$

Also we write $a\sqrt{-1} = ai$, where a represents any real number. Consistently with § 3 we write

$$\pm\sqrt{-a^2} = \pm\sqrt{a^2\cdot(-1)} = \pm\sqrt{a^2}\cdot\sqrt{-1} = \pm\, a\sqrt{-1} = \pm\, ai. \tag{II}$$

We speak of a positive or a negative imaginary according as the radical sign is preceded by a positive or a negative sign.

We define addition and subtraction of imaginaries as follows.

$$ai \pm bi = (a \pm b)\,i, \tag{III}$$

where a and b are any real numbers.

Assumption. *The commutative and associative laws of multiplication and addition of real numbers we assume to hold for imaginary numbers.*

39. Multiplication and division of imaginaries. We have already virtually defined the multiplication of imaginaries by real numbers by formula (I). Consistently with § 3 we define

$$\sqrt{-1}\cdot\sqrt{-1} = i\cdot i = i^2 = -1.$$

Thus $\sqrt{-a}\cdot\sqrt{-b} = \sqrt{a}\cdot\sqrt{b}\, i\cdot i = \sqrt{ab}\cdot(-1) = -\sqrt{ab}.$

The law of signs in multiplication may be expressed verbally as follows:

The product of two imaginaries with like signs before the radical is a negative real number. The product of two imaginaries with unlike signs is a positive real number.

For instance, $-\sqrt{-4}\cdot\sqrt{-9}=-2\cdot 3\cdot i^2=6.$

We also note that $i^2=-1,\ i^3=-i,\ i^4=1,\ i^5=i,\cdots.$

And, in general, $i^{4n+k}=i^k,\ n=0,\ 1,\ 2,\ 3,\cdots.$

We define division of imaginaries as follows:

$$\sqrt{-a}\div\sqrt{-b}=\frac{\sqrt{a}\cdot i}{\sqrt{b}\cdot i}=\sqrt{\frac{a}{b}}.$$

In operating with imaginary numbers, a number of the form $\sqrt{-a}$ should always be written in the form $\sqrt{a}\,i$ before performing the operation. This avoids temptation to the following error:

$$\sqrt{-a}\cdot\sqrt{-b}=\sqrt{(-a)\cdot(-b)}=\sqrt{ab}.$$

EXAMPLES

Simplify the following:

1. $\sqrt{-8}\cdot\sqrt{-2}.$

Solution. $\sqrt{-8}\cdot\sqrt{-2}=\sqrt{8}\cdot i\cdot\sqrt{2}\cdot i=\sqrt{2\cdot 8}\cdot i^2=4\cdot(-1)=-4.$

2. $\dfrac{1}{i^5}.$

Solution. $\dfrac{1}{i^5}=\dfrac{1}{i}=\dfrac{i}{-1}=-i.$

EXERCISES

Simplify the following:

1. $\sqrt{-49}.$

2. $\sqrt{-50}.$

3. $\sqrt{3}\,\sqrt{-48}.$

4. $\sqrt{-3}\,\sqrt{-27}.$

5. $\sqrt{-12\,a^2b^2c^2}.$

6. $\sqrt{2\,ax-(a^2+x^2)}.$

7. $\sqrt{-x^{2n}}\,\sqrt{-x^{2n+1}}.$

8. $(3\sqrt{-4})^2.$

9. $(-\sqrt{-12})^3.$

10. $\dfrac{\sqrt{-32}}{\sqrt{-12}}.$

11. $i^{13}-i^{17}.$

12. $i^{20} - i^{18}$.

13. $\dfrac{1}{i^{41}}$.

14. $\left(\dfrac{2}{i} - \dfrac{i}{2}\right)^2$.

15. $\dfrac{(-i)^3 + (-i)^2}{1 - i}$.

16. $i + 2i^2 + 3i^3 + 4i^4$.

17. $\dfrac{1}{i^{25}} + \dfrac{1}{i^{27}}$.

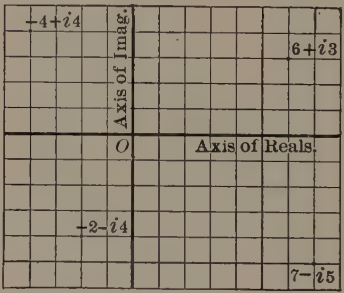
18. $\left(-\sqrt{-3\sqrt{-4}}\right)^4$.

19. $\dfrac{9\sqrt{-10}}{3\sqrt{-2}} + 9\sqrt{-10} \cdot 3\sqrt{-2}$.

20. $\dfrac{2\sqrt{-8}\sqrt{-18}}{i} + \sqrt{-6}\sqrt{-8}\sqrt{-12}$.

40. Complex numbers. The solution of the quadratic equation with negative discriminant, § 19, affords us an expression which consists of a real number connected with an imaginary number by a + or a − sign. Such an expression is called a **complex number.** It consists of two parts which are of different kinds, the real part and the imaginary part. Thus $6 + 4i$ means 6 1's + 4 i's. To any pair of real numbers (x, y) corresponds a complex number $x + iy$, and conversely.

41. Graphical representation of complex numbers. We can represent all real numbers on a single straight line. When we wished to represent two variables simultaneously (§ 17), we made use of the plane, and assumed a one-to-one correspondence between the points on the plane and the pairs of numbers (x, y). The general complex number $x + iy$ depends on the values of the independent real numbers x and y, and may then properly be represented by a point on a plane. We represent real numbers on the horizontal axis, imaginary numbers on the vertical axis, and the complex number $x + iy$ by the point (x, y) on the plane. Thus the complex numbers $6 + i3$, $-4 + i4$, $7 - i5$, $-2 - i4$ are represented by points on the plane as indicated in the figure.



42. Equality of complex numbers. We *define* the two complex numbers $a + ib$ and $c + id$ to be equal when and only when $a = c$ and $b = d$.

Symbolically, $$a + ib = c + id$$
when and only when $a = c$, and $b = d$.

The definition seems reasonable, since 1 and i are different in kind, and we should not expect any real multiple of one to cancel any real multiple of the other.

Similarly, if we took for units not abstract expressions as 1 and i, but concrete objects as trees and streets, we should say that

$$a \text{ trees} + b \text{ streets} = c \text{ trees} + d \text{ streets}$$

when and only when $a = c$ and $b = d$.

The foregoing definition may be stated in the form of the

PRINCIPLE. *When two numerical expressions involving imaginaries are equal to each other, we may equate real parts and imaginary parts separately.*

The graphical interpretation of the definition of equality of complex numbers is that equal complex numbers are represented by the same point on the plane.

From the definition given we see that $a + ib = 0$ when and only when $a = b = 0$.

ASSUMPTION. *We assume that complex numbers obey the commutative, the associative, and the distributive laws.*

This assumption includes the usual rules for the removal of parentheses.

We are now able to define the fundamental operations on complex numbers.

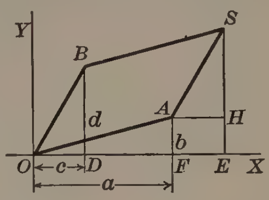
43. Addition and subtraction. By applying the assumption just made we obtain the following symbolic expression for the operations of addition and subtraction of any two complex numbers $a + ib$ and $c + id$:
$$a + ib \pm (c + id) = a \pm c + i(b \pm d). \tag{1}$$

RULE. *To add (subtract) complex numbers, add (subtract) the real and imaginary parts separately.*

44. Graphical representation of addition. We now proceed to give the graphical interpretation of addition and subtraction.

THEOREM. *The sum of two numbers $A = a + ib$ and $B = c + id$ is represented by the fourth vertex of the parallelogram formed on OA and OB as sides.*

Let OX represent the axis of reals and OY that of imaginaries. Let A and B represent $a+ib$ and $c+id$ respectively, and let $OASB$ be the completed parallelogram of which OA and OB are two of the sides. We have to prove that the coördinates of S are $a+c$ and $b+d$ (see (1)).



Draw $ES \perp OX$, $AH \perp ES$, $DB \perp OX$, $FA \perp OX$. $\triangle AHS = \triangle ODB$ since their sides are respectively parallel and $OB = AS$.

Then $\quad DB = HS = d,$

and $\quad ES = EH + HS = b + d.$

Also $\quad OD = AH = FE = c,$

and $\quad OE = OF + FE = a + c.$

Hence the point S has the coördinates $a+c$ and $b+d$, and therefore represents the sum of A and B.

EXERCISES

1. The difference $A-B$ of two numbers $A=a+ib$ and $B=c+id$ is represented by the extremity D of the line OD drawn from the origin in the direction BA, and equal to BA.

2. Represent graphically the following expressions:

(a) $3-i$.

(b) $2i+7$.

(c) $-4-2i$.

(d) $i-1$.

(e) $\frac{1}{2}+5i$.

(f) $\dfrac{4-3i}{2}$.

(g) $\dfrac{1}{1-i}$.

(h) $(2+i)+(3-2i)$.

(i) $(1-i)-(1+i)$.

(j) $2(3-4i)-4(1-2i)$.

(k) $(4i-5)+(4-5i)$.

(l) $(-\frac{1}{2}+2i)-(\frac{3}{2}-4i)$.

(m) $(-3-\frac{3}{2}i)+(1-\frac{3}{2}i)$.

(n) $3i+\dfrac{2+1}{i}$.

In the following exercises apply the principle of § 42. Find the real values of x and y satisfying the equations:

3. $(3+i)x+(1-2i)y+7i=0$.

4. $x+ixy-y=5+36i$.

5. $2x-3y+i(x^2-y^2)=4$.

6. $(3i+4)(x-y)=(i-1)x+2i(1-y)+3$.

45. Multiplication of complex numbers. The assumption of § 42 enables us to multiply complex numbers as follows:

$$\begin{array}{r} a + ib \\ c + id \\ \hline ac + icb + iad + i^2bd = ac - bd + i(cb + ad). \end{array}$$

46. Conjugate complex numbers. Complex numbers differing only in the sign of their imaginary parts are called **conjugate complex numbers.**

THEOREM. *The sum and the product of conjugate complex numbers are real numbers.*

Thus
$$a + ib + a - ib = 2a,$$
$$(a + ib)(a - ib) = a^2 + b^2.$$

47. Division of complex numbers. The quotient of two complex numbers may always be expressed as a single complex number.

RULE. *To express the quotient* $\frac{a+ib}{c+id}$ *in the form* $x + iy$, *rationalize the denominator, using as a rationalizing factor the conjugate of the denominator.*

Thus
$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id}$$
$$= \frac{ac + bd - i(ad - bc)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} - i\frac{ad - bc}{c^2 + d^2}. \qquad (1)$$

We have now defined the four fundamental operations on complex numbers and shall make frequent use of them. If the question remains, "After all, what are these so-called numbers?" we may reply: "They are expressions for which we have defined the fundamental algebraic operations. And, since they have the properties of numbers, they must be recognized as such, just as a flower which has all the characteristic properties of a known species is thereby determined to belong to that species." Furthermore, our operations have been so defined that if the imaginary parts of the complex numbers vanish and the numbers become real, the equation expressing any operation on complex numbers reduces to one expressing the same operation on the real parts of the numbers. Thus in (1) above, if $b = d = 0$, the equation reduces to
$$\frac{a}{c} = \frac{a}{c}.$$

EXAMPLES

Perform the indicated operations and simplify:

1. $(2+\sqrt{-2})(4+\sqrt{-5})$.

Solution.
$$\begin{aligned} 2+\sqrt{-2} &= 2+\sqrt{2(-1)} = 2+i\sqrt{2}\\ 4+\sqrt{-5} &= 4+\sqrt{5(-1)} = \underline{4+i\sqrt{5}}\\ &\qquad 8-\sqrt{10}+i4\sqrt{2}+i2\sqrt{5}\\ &\qquad = 8-\sqrt{10}+(4\sqrt{2}+2\sqrt{5})i. \end{aligned}$$

2. $5 \div (\sqrt{2}-i\sqrt{3})$.

Solution.
$$\frac{5}{\sqrt{2}-i\sqrt{3}} = \frac{5(\sqrt{2}+i\sqrt{3})}{(\sqrt{2}-i\sqrt{3})(\sqrt{2}+i\sqrt{3})} = \frac{5(\sqrt{2}+i\sqrt{3})}{2+3} = \sqrt{2}+i\sqrt{3}.$$

EXERCISES

Perform the indicated operations and simplify:

1. $\dfrac{2-2i}{1+i}$.
2. $\dfrac{4}{1+\sqrt{-3}}$.
3. $\dfrac{3}{\sqrt{2}+i}$.
4. $(\sqrt{3}+i\sqrt{2})(\sqrt{2}+i\sqrt{3})$.
5. $(a\sqrt{b}+ic\sqrt{d})(a\sqrt{b}-ic\sqrt{d})$.
6. $(\sqrt{1+i}+\sqrt{1-i})^2$.
7. $(x+iy)^3$.
8. $(3+i)^4+(3-i)^4$.
9. $\dfrac{1}{(1-i)^2}-\dfrac{1}{(1+i)^2}$.
10. $\dfrac{(1+i)^2}{1-i}$.
11. $\dfrac{87}{4+7\sqrt{-5}}$.
12. $(-1+i\sqrt{3})^2+(-1-i\sqrt{3})^2$.
13. $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3$.
14. $\left(\dfrac{1+i}{\sqrt{2}}\right)^4$.
15. $\dfrac{\sqrt{3}+i\sqrt{2}}{\sqrt{3}-i\sqrt{2}}$.
16. $\dfrac{1+2i+3i^2}{1-2i+3i^2}$.
17. $\left(\dfrac{2-i}{2+i}\right)^2-\left(\dfrac{2+i}{2-i}\right)^2$.
18. $\dfrac{1+i+2i^2+3i^3}{i+2i^2+3i^3+4i^4}$.
19. $\dfrac{8}{(i+1)^3(i-1)^3}$.
20. $\dfrac{\sqrt{-a}-\sqrt{-b}}{\sqrt{-a}+\sqrt{-b}}$.

Prove the following relations:

21. $(2+i)^2 = \dfrac{7+i}{1-i}$.

22. $(1-2i)^2 = \dfrac{11-2i}{2i-1}$.

23. $\dfrac{1}{2}(3+5i)^2 = \dfrac{36i-77}{3i+4}$.

24. $(\sqrt{3}+\sqrt{2}\,i)^2 = \dfrac{25}{1-\sqrt{-24}}$.

25. $(1-i)^3 = \dfrac{-4i}{1+i}$.

26. $\dfrac{(1+i)^2}{\sqrt{3}+i} = \dfrac{\sqrt{3}-i}{(1-i)^2}$.

Perform the indicated operations and simplify:

27. $\dfrac{a+ib}{a-ib} - \dfrac{a-ib}{a+ib}$.

28. $\dfrac{a+ib}{c+id} + \dfrac{a-ib}{c-id}$.

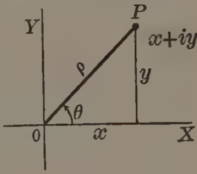
29. $\dfrac{a+i\sqrt{1-a^2}}{a-i\sqrt{1-a^2}}$.

30. $\dfrac{\sqrt{1+x}+i\sqrt{1-x}}{\sqrt{1+x}-i\sqrt{1-x}} - \dfrac{\sqrt{1-x}+i\sqrt{1+x}}{\sqrt{1-x}-i\sqrt{1+x}}$.

31. Reduce $\dfrac{12-5i}{2-3i}$ and $\dfrac{-5}{1+2i}$ to simplest form and represent their sum and their difference graphically.

32. Reduce $\dfrac{5-11i}{2(1-i)}$ and $\dfrac{-(7+9i)}{2(1+i)}$ to simplest form and represent their sum and their difference graphically.

48. Polar representation. The graphical representation of complex numbers given in § 41 suggests the graphical interpretation of the operations of addition and subtraction given in § 44. But the graphical meaning of the operations of multiplication and division may be shown more clearly in another manner. We have seen that we may represent $x+iy$ by the point $P(x, y)$ on the plane. Let us represent the angle between OP and the axis of reals by θ (read *theta*). This is called the **angle** of the complex number $x+iy$. We will denote the length of the line OP by ρ (read *rho*). This is called the **modulus** of $x+iy$. Then from the figure



$$x = \rho \cos\theta, \tag{1}$$

$$y = \rho \sin\theta, \tag{2}$$

and
$$x^2 + y^2 = \rho^2. \tag{3}$$

Hence the complex number $x + iy$ may be written in the form

$$x + iy = \rho(\cos\theta + i\sin\theta), \tag{4}$$

where the relations between x, y and ρ, θ are given by (1), (2), and (3). When the value of ρ is found by the use of (3), the positive sign is always taken. A number expressed in this way is in **polar form,** and may be designated by (ρ, θ). We observe that a complex number lies on a circle whose center is the origin and whose radius is the modulus of the number. The angle is the one which the line representing the modulus makes with the axis of real numbers.

When the values of ρ and θ are given we can find the values of x and y for the corresponding complex number by means of (1) and (2). When a number is given in polar form it should be kept in mind that the modulus, or the value of ρ, is the coefficient of the expression $\cos\theta + i\sin\theta$.

Thus in the number $2(\cos 30° + i\sin 30°)$, 2 is the modulus and 30° is the angle.

EXAMPLE

Find the modulus and the angle of the number $1 + i\sqrt{3}$ and write the number in polar form.

Solution. Let $1 + i\sqrt{3} = x + iy$; then $x = 1$, $y = \sqrt{3}$.

By (3), $\rho = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2.$

By (1), $x = \rho\cos\theta$, or $\cos\theta = \dfrac{x}{\rho} = \dfrac{1}{2}.$

Hence θ must equal either 60° or 300°. But since the number $1 + i\sqrt{3}$ is represented in the first quadrant, the only possible value is $\theta = 60°$, and we have

$$1 + i\sqrt{3} = 2(\cos 60° + i\sin 60°).$$

EXERCISES

Find the modulus and the angle of each of the following numbers and write them in polar form. Plot the numbers.

1. $-1 + i\sqrt{3}.$
2. $1 - i.$
3. $-3 - 3i.$
4. $\sqrt{3} + i.$
5. $-\sqrt{8} + i\sqrt{8}.$
6. $\frac{1}{2} - \frac{1}{2}\sqrt{3}\,i.$
7. $-3 - 4i.$
8. $5 + 12i.$

Change the following complex numbers from the polar form to the form $x + iy$. Plot the numbers.

9. $\cos 225° + i \sin 225°$.

10. $2(\cos 300° + i \sin 300°)$.

11. $6(\cos 120° + i \sin 120°)$.

12. $2\sqrt{2}(\cos 135° + i \sin 135°)$.

13. $\frac{5}{2}(\cos 180° + i \sin 180°)$.

14. $\cos 270° + i \sin 270°$.

49. Multiplication in polar form. If we have two numbers $\rho(\cos\theta + i\sin\theta)$ and $\rho'(\cos\theta' + i\sin\theta')$, we may multiply them as follows:

$$\rho(\cos\theta + i\sin\theta)\,\rho'(\cos\theta' + i\sin\theta')$$

$$= \rho\rho'(\cos\theta\cos\theta' + i\cos\theta\sin\theta' + i\sin\theta\cos\theta' + i^2\sin\theta\sin\theta')$$

collecting terms,

$$= \rho\rho'[(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')]$$

by the addition theorem in trigonometry,

$$= \rho\rho'[\cos(\theta + \theta') + i\sin(\theta + \theta')]. \tag{1}$$

In this product $\rho\rho'$ is the new modulus and $\theta + \theta'$ the new angle. We may now state the following

THEOREM. *The product of the two numbers $\rho(\cos\theta + i\sin\theta)$ and $\rho'(\cos\theta' + i\sin\theta')$ has as its modulus $\rho\rho'$ and as its angle $\theta + \theta'$.*

It is observed that the product of two numbers is represented on a circle whose radius is the product of the radii of the circles on which the factors are represented. The angle of the product is the sum of the angles of the factors.

50. Powers of numbers in polar form. When the two factors of the preceding section, (ρ, θ) and (ρ', θ'), are equal, that is, when $\rho = \rho'$ and $\theta = \theta'$, equation (1) assumes the form

$$[\rho(\cos\theta + i\sin\theta)]^2 = \rho^2(\cos 2\theta + i\sin 2\theta). \tag{1}$$

This suggests as a form for the nth power of a complex number

$$[\rho(\cos\theta + i\sin\theta)]^n = \rho^n(\cos n\theta + i\sin n\theta). \tag{2}$$

The theorem expressed by (2) is known as **De Moivre's** theorem. Stated verbally it is as follows: The modulus of the nth power of a number is the nth power of the modulus of the number. The angle of the nth power of a number is n times the angle of the number.

51. Division in polar form. If we have, as before, two complex numbers in polar form, (ρ, θ) and (ρ', θ'), we may obtain their quotient as follows:

$$\frac{\rho(\cos\theta + i\sin\theta)}{\rho'(\cos\theta' + i\sin\theta')}$$

(rationalizing)
$$= \frac{\rho(\cos\theta + i\sin\theta)}{\rho'(\cos\theta' + i\sin\theta')} \cdot \frac{(\cos\theta' - i\sin\theta')}{(\cos\theta' - i\sin\theta')}$$

$$= \frac{\rho[\cos(\theta - \theta') + i\sin(\theta - \theta')]}{\rho'(\cos^2\theta' + \sin^2\theta')}$$

(since $\sin^2\theta' + \cos^2\theta' = 1$)
$$= \frac{\rho}{\rho'}[\cos(\theta - \theta') + i\sin(\theta - \theta')].$$

We may now state the following

THEOREM. *The quotient of two complex numbers has as its modulus the quotient of the moduli of the factors, and as its angle the difference of the angles of the factors.*

EXAMPLES

1. Find the moduli and angles of the numbers $2 - 2i$ and $\sqrt{3} + i$ and of their product. Plot the three numbers.

Solution. Let $2 - 2i = x + iy$.

Then $$x = 2, \quad y = -2,$$
and $$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = 2\sqrt{2}.$$

$$\cos\theta = \frac{x}{\rho} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

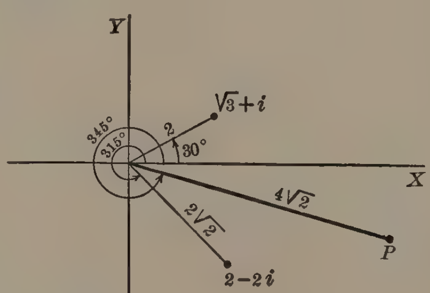
Hence $\theta = 45°$ or $315°$.

But since $2 - 2i$ is represented in the fourth quadrant,

$$\theta = 315°.$$

Let $\sqrt{3} + i = x' + iy'$.

Then $$x' = \sqrt{3}, \quad y' = 1,$$
and $$\rho' = \sqrt{x'^2 + y'^2} = \sqrt{3 + 1} = 2.$$



$$\cos\theta' = \frac{x'}{\rho'} = \frac{\sqrt{3}}{2}.$$

Hence $\theta' = 30°$ or $330°$.

But since $\sqrt{3} + i$ is represented in the first quadrant, $\theta' = 30°$.

By the theorem of § 49 the modulus of the product is

$$\rho\rho' = 2\sqrt{2} \cdot 2 = 4\sqrt{2}.$$

The angle of the product is $\theta + \theta' = 315° + 30° = 345°$.

Hence $$P = (2 - 2i)(\sqrt{3} + i) = 4\sqrt{2}(\cos 345° + i\sin 345°).$$

2. Find the moduli and angles of the numbers $2-2\sqrt{3}\,i$ and $1+i$ and of their quotient. Plot the three numbers.

Solution. Let $2-2\sqrt{3}\,i = x+iy$.

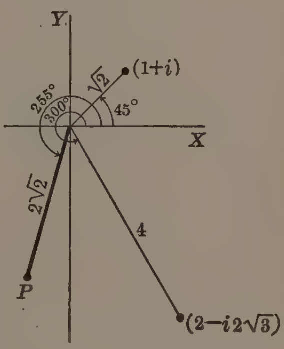
Then $x=2,\quad y=-2\sqrt{3},$

and $\rho=\sqrt{x^2+y^2}=\sqrt{4+12}=4.$

$$\cos\theta=\frac{x}{\rho}=\frac{2}{4}=\frac{1}{2}.$$

Hence $\theta=60°$ or $300°$.

But since $2-2\sqrt{3}\,i$ is represented in the fourth quadrant, $\theta=300°$.



Let $1+i=x'+iy'$.

Then $x'=1,\quad y'=1,$

and $\rho'=\sqrt{x'^2+y'^2}=\sqrt{1+1}=\sqrt{2}.$

$$\cos\theta'=\frac{x'}{\rho'}=\frac{1}{\sqrt{2}}.$$

Hence $\theta'=45°$ or $315°$.

But since $1+i$ is represented in the first quadrant, $\theta'=45°$.

By the theorem, § 51, the modulus of the quotient is $\dfrac{\rho}{\rho'}=\dfrac{4}{\sqrt{2}}=2\sqrt{2}.$

The angle of the quotient is $\theta-\theta'=300°-45°=255°$.

Hence $P=\dfrac{2-2\sqrt{3}\,i}{1+i}=2\sqrt{2}\,(\cos 255°+i\sin 255°).$

EXERCISES

Find the moduli and angles of the following numbers and of the indicated products, quotients, or powers. Plot the numbers in each case.

1. $(2+2i)(-1+\sqrt{3}\cdot i)$.
2. $(-\sqrt{3}+i)(-1-i)$.
3. $(\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i)(\frac{1}{2}-\frac{1}{2}i)$.
4. $6i\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right)$.
5. $(3-3i)(2+i\sqrt{12})$.
6. $(4+3i)(1+\frac{12}{5}i)$.
7. $(1+i)^8$.
8. $\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^2$.
9. $(-3-\sqrt{3}\,i)^3$.
10. $(-2+2i)^4$.
11. $\dfrac{\sqrt{3}-i}{-\sqrt{2}-\sqrt{2}\,i}$.
12. $\dfrac{2-2\sqrt{3}\,i}{-i}$.
13. $\dfrac{-3-3\sqrt{3}\cdot i}{2+2i}$.
14. $\dfrac{-2}{3i}$.
15. $\dfrac{-7+24i}{3+4i}$.
16. $(-1+i)^{10}$.
17. $(3+3i)(-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i)(-2-2i)$.

18. $\dfrac{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2}{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}$.

19. $[2(\cos 60° + i \sin 60°)]^3$.

20. $[\frac{1}{2}(\cos 15° + i \sin 15°)]^2$.

21. $(\cos 45° + i \sin 45°)^{16}$.

22. $[3(\cos 75° + i \sin 75°)]\,[\frac{2}{3}(\cos 15° + i \sin 15°)]$.

23. $\dfrac{\frac{3}{4}(\cos 180° + i \sin 180°)}{\frac{1}{2}(\cos 100° + i \sin 100°)}$.

24. $\dfrac{[2(\cos 135° + i \sin 135°)]^4}{\left[\frac{4\sqrt{5}}{5}(\cos 315° + i \sin 315°)\right]^2}$.

25. For what values of n will $(1+\sqrt{-3})^n$ be a real number?

52. Roots of complex numbers. We have seen that the square of a complex number has as its modulus the square of the modulus of the number, and as its angle twice the original angle.

Thus the number $(1, 30°)$, or $1 \cdot (\cos 30° + i \sin 30°)$, has as its square the number $(1, 60°)$. Also the number $(1, 210°)$ has as its square $(1, 420°)$. But $(1, 420°) = (1, 60°)$; for if two complex numbers have the same modulus and their angles differ by a multiple of 360°, they are represented graphically by the same point, and are therefore identical. For example, $(1, 60°) = (1, 420°) = (1, 780°)$, etc. Hence it appears that any complex number with an angle greater than 360° is equivalent to one with an angle less than 360°.

We have, then, found two numbers $(1, 30°)$ and $(1, 210°)$ each of whose squares equals the number $(1, 60°)$, that is, we have found the square roots of this number.

In general, the modulus of the square root of a number is the positive square root of the modulus of the number. The angle of the square root of a number is one such that if we double it we get either the original angle or one which differs from it by 360°. Expressed in symbols, if the original angle is θ, the angles of the square roots are

$$\frac{\theta}{2} \text{ and } \frac{\theta}{2} + 180°.$$

This may be denoted more compactly as follows:

$$\frac{\theta}{2} + k\,180°, \quad (k = 0, 1),$$

by which we mean that we substitute in the expression indicated, first, the value $k = 0$, and then the value $k = 1$, obtaining the same values $\frac{\theta}{2}$ and $\frac{\theta}{2} + 180°$, for the angles which were given above.

In a similar manner, if we seek all the numbers with angles less than 360° which cubed give a certain number, we must find three angles which multiplied by 3 give the original angle or one which differs from it by a multiple of 360°.

For example, the three numbers $(1, 20°)$, $(1, 140°)$, and $(1, 260°)$ have as their cubes the three numbers $(1, 60°)$, $(1, 420°)$, and $(1, 780°)$ respectively. But since all of these numbers have the same modulus and their angles differ by either 360° or 720°, they are really the same number; that is, the numbers first given are the three cube roots of the number $(1, 60°)$.

In general, the modulus of a cube root of a complex number is the real cube root of the modulus of the number. The angle of a cube root is an angle such that if we multiply it by 3, we obtain either the original angle or one which differs from it by a multiple of 360°. If the original angle is denoted by θ, the three angles of the cube roots are

$$\frac{\theta}{3} + k\,120°, \quad (k = 0, 1, 2).$$

We may treat the problem of finding the nth roots of a number (ρ, θ) similarly. The modulus of the nth root of (ρ, θ) is the real positive nth root of ρ, namely $\sqrt[n]{\rho}$. The angles are those angles which, multiplied by n, give either θ or an angle which differs from θ by a multiple of 360°. There are n such angles less than 360°.

In the notation which we have adopted the angles of the nth roots are

$$\frac{\theta}{n} + k \cdot \frac{360}{n}, \quad (k = 0, 1, 2, \cdots, n-1).$$

Thus $$\sqrt[n]{\rho(\cos\theta + i\sin\theta)} = \sqrt[n]{\rho}\left[\cos\left(\frac{\theta}{n} + k\cdot\frac{360}{n}\right) + i\sin\left(\frac{\theta}{n} + k\cdot\frac{360}{n}\right)\right],$$

where for a given value of n, k takes on the values $0, 1, \cdots, n-1$, and where $\sqrt[n]{\rho}$ indicates the real positive nth root of ρ.

For example, the five angles of the fifth roots of a number whose angle is 60° are obtained by adding to $\frac{60°}{5} = 12°$, the angles $k \cdot \frac{360}{5}$, where $k = 0, 1, 2, 3, 4$; that is, the angles are $12°$, $12° + 72° = 84°$, $12° + 2 \cdot 72° = 156°$, $12° + 3 \cdot 72° = 228°$, $12° + 4 \cdot 72° = 300°$.

When, in the following exercises, the radical sign, $\sqrt[n]{}$, is used over a complex number, or over a real number which is thought of as a complex number with zero imaginary part, all n of the roots are meant. When only the arithmetical square root of a real number is intended, the usage explained on page 5 is followed. The context will make it clear in each case which meaning of the radical is to be taken.

EXAMPLES

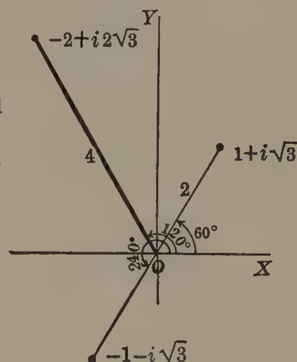
1. Perform the indicated operation and plot:

$$\sqrt{-2+2\sqrt{3}\,i}.$$

Solution. We first express the number $-2+2\sqrt{3}i$ in polar form.

$$\rho=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+12}=4.$$

$$\cos\theta=\frac{-2}{4}=-\frac{1}{2},\quad \theta=120^\circ \text{ or } 240^\circ.$$



But since $-2+2\sqrt{3}\,i$ is in the second quadrant, $\theta=120^\circ.$

Hence $-2+2\sqrt{3}i=4(\cos 120^\circ+i\sin 120^\circ).$

Now $\sqrt{\rho}=\sqrt{4}=2,$

which is the modulus of the square roots.

The angles are

$$\frac{\theta}{2}=\frac{120^\circ}{2}=60^\circ,$$

and $$\frac{\theta}{2}+180^\circ=\frac{120^\circ}{2}+180^\circ=240^\circ.$$

Hence $\sqrt{-2+\sqrt{3}i}=2(\cos 60^\circ+i\sin 60^\circ)=1+\sqrt{3}i,$

or $\qquad 2(\cos 240^\circ+i\sin 240^\circ)=-1-\sqrt{3}i.$

It is sometimes more convenient to find the roots of a complex number in another way. For example,

2. Find $\sqrt{3+4\,i}.$

Solution. We see from § 52 that a root of a complex number is always a complex number.

Hence we may write $\qquad \sqrt{3+4i}=x+iy.$

Squaring, $\qquad 3+4i=x^2+2ixy-y^2.$

Then, by § 42, $\qquad x^2-y^2=3, \qquad (1)$

$\qquad 2xy=4. \qquad (2)$

Squaring (1) and (2), $\qquad x^4-2x^2y^2+y^4=9$

$\qquad 4x^2y^2=16$

Adding, $\qquad x^4+2x^2y^2+y^4=25$

Hence $\qquad x^2+y^2=\pm 5.$

But since x and y are real, the sum of their squares must be positive. Then we must take

$$x^2+y^2=5$$

Adding (1), $\qquad x^2-y^2=3$

we obtain $\qquad 2x^2=8$

$$x^2=4,$$

$$x=\pm 2.$$

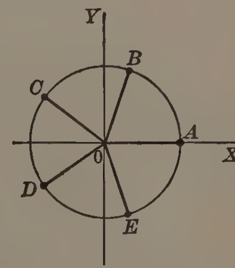
Substituting this value of x in the equation $2xy=4$, we find $y=\pm 1.$

$$\therefore \sqrt{3+4i}=2+i \quad\text{or}\quad -2-i.$$

3. Solve the equation $x^5-1=0$, and represent the roots graphically.

Solution. $x^5=1$, or $x=\sqrt[5]{1}$.

Let $1=1+0\cdot i=\rho(\cos\theta+i\sin\theta)$. Then $\rho=1$, $\theta=0°$.



$$x=\sqrt[5]{1(\cos 0°+i\sin 0°)}$$

$$=\sqrt[5]{1}\left[\cos\left(\frac{0°}{5}+k\cdot\frac{360°}{5}\right)+i\sin\left(\frac{0°}{5}+k\cdot\frac{360°}{5}\right)\right]$$

(where k takes on the values 0, 1, 2, 3, 4)

$$=\begin{cases}\cos\ \ 0°+i\sin 0°=1, \text{ when } k=0,\\ \cos\ 72°+i\sin 72°, \text{ when } k=1,\\ \cos 144°+i\sin 144°, \text{ when } k=2,\\ \cos 216°+i\sin 216°, \text{ when } k=3,\\ \cos 288°+i\sin 288°, \text{ when } k=4.\end{cases}$$

These numbers we observe lie on a circle of unit radius at the vertices of a regular pentagon.

EXERCISES

In the following exercises perform the indicated operations and represent graphically the complex number and its roots:

1. Work example 2 above, using polar representation.

2. $\sqrt[3]{4+4\sqrt{3}\,i}$.

3. $\sqrt{i}$.

4. $\sqrt{-i}$.

5. Show that $\sqrt{i}+\sqrt{-i}=\pm\sqrt{2}$, or $\pm i\sqrt{2}$.

6. $\sqrt{-1+2\sqrt{2}\,i}$.

7. $\sqrt[3]{3+3i}$.

8. $\sqrt{\sqrt{3}-i}$.

9. $\sqrt[5]{4-4i}$.

10. $\sqrt{5+12i}$.

11. $\sqrt{-1+4\sqrt{-5}}$.

12. $\sqrt[3]{8(\cos 15°+i\sin 15°)}$.

13. $\sqrt[4]{16(\cos 200°+i\sin 200°)}$.

14. $\sqrt[6]{8(\cos 60°+i\sin 60°)}$.

15. $\sqrt[5]{\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$.

16. $\sqrt[4]{9i}$.

17. $\sqrt[3]{8}$.

18. $\sqrt[3]{-128}$.

Solve the following equations and illustrate the results graphically:

19. $x^5-32=0$.

20. $x^3-1=0$.

21. $x^3+1=0$.

22. $x^4-16=0$.

23. $x^4+1=0$.

24. $x^6-1=0$.

25. $x^8-1=0$.

26. $x^5+1=0$.

27. Show that either of the complex roots of the equation $x^3 = 1$ is the square of the other, and that the sum of the three roots is zero. Represent the three roots and their sum graphically.

28. Show graphically that the sum of the roots of the equation $x^5 = 1$ is zero.

29. Show graphically that the sum of the roots of the equation $x^n = 1$ is zero if n is a positive even integer.

NOTE. This is true when n is any positive integer, whether even or odd, as we shall see in the next chapter, § 62.

30. Show that the product of the three cube roots of 1 is 1.

31. Prove that the product of the n nth roots of 1 is 1, if n is odd, and -1 if n is even.

CHAPTER VI

THEORY OF EQUATIONS

53. Introduction. As a preliminary to the development of the methods and theorems of this chapter, a few definitions are necessary.

A term is **rational** if it may be obtained in its simplest form from unity and the letters concerned by means of the four operations of addition, subtraction, multiplication, and division, without the extraction of any root. If each of the terms of an algebraic function is rational, it is called a **rational function.**

The functions $\frac{a}{x}+\frac{b}{x^2},\ \frac{1}{x^2}-\frac{4x}{5},\ \frac{8a-bx^2}{cx^3-1}+\frac{ax^2+bx+c}{cx-f},$
are each rational in x.

A term is **integral** if the letter which is taken as the variable does not appear in the denominator. A term may be integral and still involve a radical sign. If each of the terms of an algebraic expression is integral, it is called an **integral function.**

The functions $\frac{4x}{a},\ 8x^2-\sqrt{x},\ \ ax^2+bx+c,$
are each integral in x.

An integral function is not necessarily rational, nor is every rational function integral. Thus $\frac{x^2}{4}+\sqrt{x}+8$ is integral but not rational, while $\frac{x^2}{4}+\frac{1}{x}+8$ is rational but not integral.

An algebraic function is **rational** and **integral** if each of its terms is rational and integral. Such an expression is frequently called a **polynomial.** The polynomial $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$, where $a_0, a_1, a_2, \cdots, a_n$, are all integers, n is a positive integer, and $a_0 \neq 0$, we shall call the general polynomial of the nth degree, or the polynomial in a-form.

The equation
$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad (A)$$
we shall call the general equation of the nth degree, or the equation in **a-form.** The subject of our study in this chapter is the rational, integral equation of degree n in one variable.

It is necessary to keep in mind that the symbols $a_0, a_1, \cdots, a_n$, stand for numbers. Since they are all coefficients in the same expression, we denote them all by the same letter, a; but since they are the coefficients of different terms, they must be distinguished from each other in some way. This is done by giving each a a subscript equal to one less than the number of the term in which it is used. In this way we know that a_3 is the coefficient in the fourth term, a_2 that in the third term, and so on.

The notation $f(x)$, read "f of x," is simply a symbol denoting that the expression in question is a function of x. Other letters are sometimes used to denote functions of x, as, for instance, $F(x)$, $\phi(x)$, and $Q(x)$.

If in $f(x)$ the variable x is replaced, for example, by the number 3, the resulting expression is denoted by $f(3)$. We may similarly replace x all through $f(x)$ by a letter, as, for instance, c. The resulting expression $a_0c^n + a_1c^{n-1} + \cdots + a_n$ we denote by $f(c)$.

For example, if $f(x) = 2x^3 - 3x^2 - 7x + 5$, $f(2) = 2(2)^3 - 3(2)^2 - 7 \cdot 2 + 5 = -5$, while $f(0) = 5$ and $f(-1) = 7$.

The equation $$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0, \qquad (P)$$
where $p_1, p_2, \cdots, p_n$, are all rational numbers, and n is a positive integer, we shall call the **p-form** of the equation of the nth degree. It is observed that any rational integral equation with rational coefficients may be brought into the a-form, (A), by transposition and multiplication by the least common denominator of the coefficients.

Furthermore, any general equation may be put into the p-form by dividing by a_0. Since a_0 is assumed to be different from zero, this can always be done.

For example, the equation $\frac{2}{3}x^4 - \frac{7}{8}x = \frac{4}{3}x^2 - x^4 + \frac{1}{4}$, after transposition and multiplication by 24, becomes

$$40x^4 - 32x^2 - 21x - 6 = 0,$$

which is in the a-form. Dividing by 40, we have the equation

$$x^4 - \tfrac{4}{5}x^2 - \tfrac{21}{40}x - \tfrac{3}{20} = 0,$$

which is in the p-form.

EXERCISES

Reduce the following equations to the a-form and to the p-form:

1. $x - \dfrac{x^5}{2} + \dfrac{x^3}{3} = 4 - \dfrac{x}{4} + \dfrac{3x^3}{4}.$

2. $\dfrac{3x^6}{5} + 1 = \dfrac{2x^3}{15} + 4.$

3. $.5x - .75x^2 + .25 = x^3.$

4. $1.4x^4 = 2.8x^5 - .7x + 2.1x^2$.

5. If $f(x) = 2x^3 - 4x + 6$, find $f(0), f(1), f(-2), f(a)$, and $f(-x)$.

6. If
$$f(x) = x^2 + 3x - 3,$$
find
$$f(3),\ f(c),\ f(x+1),\ f(a-1).$$

7. If
$$f(x) = 2x^4 - x^2,$$
find
$$f(\sqrt{2}),\ f(0),\ f\left(\frac{c}{2}\right);\ f\left(-\frac{1}{2}\right).$$

8. Reducing the equation $\frac{x^2}{3} + \frac{x}{6} - 1 = 0$ to the a-form and to the p-form, we get
$$2x^2 + x - 6 = 0, \tag{A}$$
and
$$x^2 + \frac{x}{2} - 3 = 0. \tag{P}$$

Let
$$f_1(x) = \frac{x^2}{3} + \frac{x}{6} - 1,$$
$$f_2(x) = 2x^2 + x - 6,$$
$$f_3(x) = x^2 + \frac{x}{2} - 3.$$

(a) Graph the three functions $f_1(x)$, $f_2(x)$, and $f_3(x)$.

(b) What relation do you notice between these graphs?

(c) What are the roots of the three equations $f_1(x) = 0$, $f_2(x) = 0$, and $f_3(x) = 0$?

(d) Are the roots of an equation changed by reducing it to the a-form and to the p-form? Why?

9. Reduce the equation $.75x^2 + 5.875x = 1$ to the a-form and find its roots.

10. If $f(x) = a_0x^2 + a_1x + a_2$, determine a_0, a_1, and a_2, so that $f(1) = 4$, $f(2) = 12$, $f(-1) = 6$.

11. If $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, determine a_0, a_1, a_2, and a_3, so that $f(0) = f(\frac{1}{2}) = 2$, $f(-\frac{1}{2}) = \frac{7}{4}$, $f(2) = 8$.

12. Given $f(x) = a_0x^2 + a_1x + a_2$, $a_0 \neq a_1$, and $f(a_0) = f(a_1)$ Express a_1 in terms of a_0.

13. Find a polynomial of the second degree which has the value 0 when $x = -1$ or 2, and the value -2 when $x = 0$. Graph the polynomial. Is it always possible to find the polynomial however we select the three values of x and the corresponding values which the polynomial is to have?

14. Find a polynomial of the second degree whose graph passes through the point (2, 4), and such that the equation formed by equating it to zero has roots -2 and 4.

15. Find a polynomial of the third degree which vanishes when $x = 0$, 1, or 2, and which equals 2 when $x = 3$.

16. Find a polynomial of the fourth degree which equals 1 when x is 0, equals 0 when x is 1 or -1, and equals 21 when x is 2 or -2.

54. Remainder Theorem. The following theorem lies at the basis of most of the work of the present chapter:

THEOREM. *If $f(x)$ is divided by $x - c$, the remainder is $f(c)$.*

Illustration. Let $f(x) = 2x^3 + 3x^2 - 4x - 6$, and let $c = 2$. The theorem will be verified for this case if the remainder obtained by dividing $f(x)$ by $x - 2$ is $f(2) = 2 \cdot 2^3 + 3 \cdot 2^2 - 4 \cdot 2 - 6 = 14$. If the division is actually performed, the remainder is found to be 14. The result of the division may be expressed thus:

$$\frac{2x^3 + 3x^2 - 4x - 6}{x - 2} = 2x^2 + 7x + 10 + \frac{14}{x - 2}$$

Proof. If $f(x)$ is divided by $x - c$, let us call the quotient $Q(x)$. We must prove that the remainder is $f(c)$, that is, that

$$\frac{f(x)}{x - c} = Q(x) + \frac{f(c)}{x - c}.$$

Consider the expressions

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

and

$$f(c) = a_0c^n + a_1c^{n-1} + \cdots + a_{n-1}c + a_n.$$

Subtracting, we get

$$\begin{aligned} f(x) - f(c) &\\ = a_0x^n + a_1x^{n-1} &+ \cdots + a_{n-1}x + a_n - (a_0c^n + a_1c^{n-1} + \cdots + a_{n-1}c + a_n) \\ = a_0(x^n - c^n) &+ a_1(x^{n-1} - c^{n-1}) + \cdots + a_{n-1}(x - c). \end{aligned}$$

But since $x - c$ is a factor of each term in the right member (type form 7, p. 2), we may take $x - c$ outside a parenthesis, and call what remains inside $Q(x)$. We have then

$$f(x) - f(c) = (x - c)Q(x), \qquad (1)$$

or, after transposing $f(c)$ and dividing by $x - c$,

$$\frac{f(x)}{x - c} = Q(x) + \frac{f(c)}{x - c},$$

which was to be proved.

Factor Theorem. *If c is a root of $f(x)=0$, then $x-c$ is a factor of $f(x)$.*

If c is a root of $f(x)=0$, then $f(c)=0$, and from (1) we have

$$f(x)=(x-c)Q(x).$$

That is, $f(x)$ is expressed in factored form, with $x-c$ as one of the factors.

EXERCISES

1. State and prove the converse of the Factor Theorem.

2. By use of the Remainder Theorem, find the remainder when $2x^4+x^3-6x^2+1$ is divided by $x-1$; by $x+2$.

3. By use of the Remainder Theorem, find the remainder when $3x^4+2x^2-1$ is divided by $x-\frac{1}{2}$; by x.

4. By use of the Remainder Theorem, find the remainder when $2x^5-x^3-x^2+4x-1$ is divided by $x-3$; by $x+3$.

5. By use of the Remainder Theorem, find the remainder when x^7+1 is divided by $x+1$; by $x-1$.

6. By use of the Remainder Theorem, find the remainder when $x^{13}+a^{13}$ is divided by $x+a$; by $x-a$.

7. Show that 2 is a root of the equation $2x^3-3x^2-4x+4=0$.

8. Show that if $f(c)\neq 0$, then $f(x)$ is not divisible by $x-c$.

9. Find a polynomial $f(x)$ of the second degree such that 1 and 2 are roots of $f(x)=0$, and $f(x)$ has the value 8 when $x=0$.

10. Find a polynomial $f(x)$ of the third degree such that 0, -1, and 3 are roots of $f(x)=0$, and $f(x)$ has the value 12 when $x=1$.

55. Synthetic division. In plotting the function

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n,$$

where the a's are integers, by the method of § 17, it is necessary to find the values of the function for various values of x; that is, we must obtain the values of $f(1)$, $f(2)$, $f(3)$, etc. The Remainder Theorem tells us that these are nothing else than the values of the remainders after dividing $f(x)$ by $x-1$, $x-2$, $x-3$, etc., which we may find rapidly if we make use of the following abridged method of division.

The method may be illustrated by the following example:

Let $$f(x) = 2x^4 - 3x^3 + x^2 - x - 9, \quad c = 2,$$

and let us divide $f(x)$ by $x - 2$.

By long division we have

$$
\begin{array}{rrrrr|l}
\mathbf{2}x^4 - & 3x^3 + & x^2 - & x - & 9 & x - 2 \\
\cline{6-6}
2x^4 - & 4x^3 & & & & 2x^3 + x^2 + 3x + 5 \\
\cline{1-2}
 & \mathbf{1}x^3 + & x^2 & & & \\
 & \mathbf{1}x^3 - & 2x^2 & & & \\
\cline{2-3}
 & & \mathbf{3}x^2 - & x & & \\
 & & 3x^2 - & 6x & & \\
\cline{3-4}
 & & & \mathbf{5}x - & 9 & \\
 & & & 5x - & 10 & \\
\cline{4-5}
 & & & + & \mathbf{1} & \\
\end{array}
$$

We can abbreviate this process by observing the following facts. Since x may be regarded as merely the carrier of the coefficient, we may omit writing it. Also we need not rewrite the first number of the partial product, as it is only a repetition of the number directly above it in full-faced type. Our process now assumes the form

$$
\begin{array}{rrrrr|l}
\mathbf{2} & -3 & +1 & -1 & -9 & 1 - 2 \\
\cline{6-6}
 & -4 & & & & 2 + 1 + 3 + 5 \\
\cline{1-2}
 & \mathbf{+1} & & & & \\
 & & -2 & & & \\
\cline{3-3}
 & & \mathbf{+3} & & & \\
 & & & -6 & & \\
\cline{3-4}
 & & & \mathbf{+5} & & \\
 & & & & -10 & \\
\cline{4-5}
 & & & & +\ \mathbf{1} & \\
\end{array}
$$

Since the minus sign of the 2 changes every sign in forming the partial product, if we replace -2 by $+2$ we may add the partial product to the number in the dividend instead of subtracting. This is also desirable, since the number which we are substituting for x is 2, not -2. Thus, bringing all our figures on one line, dropping the 1 in the divisor, replacing -2 by 2, and omitting the coefficients of the quotient, we have

$$
\begin{array}{rrrrr|l}
2 & -3 & +1 & -1 & -9 & \underline{2} \\
 & +4 & +2 & +6 & +10 & \\
\hline
2 & +1 & +3 & +5 & +1 & \\
\end{array}
$$

The last number in the lowest line is the remainder in the division. Hence it is the value of the function $f(x)$ when x is replaced by 2, that is, $f(2)$.

It is also to be observed that, from the nature of the operation, the numbers preceding the remainder in the last line are the coefficients of the quotient in the division. In this case the quotient is

$$2x^3 + x^2 + 3x + 5.$$

We have illustrated the following rule. Since the process may be looked upon as merely a convenient arrangement of the operation of long division, no formal proof will be given.

RULE FOR SYNTHETIC DIVISION. *Write the coefficients of the polynomial in order, supplying* 0 *when a term is lacking.*

Multiply the number to be substituted for x by the first coefficient, and add (algebraically) the product to the second coefficient.

Multiply this sum by the number to be substituted for x, add to the third coefficient, and proceed until all the coefficients are used. The last sum obtained is the remainder and also the value of the polynomial when the number is substituted for the variable.

The method of synthetic division is useful not only in finding the values of the function for purposes of plotting, but also in determining whether the function has any factors of the form $x - c$. For if by the process of synthetic division the remainder $f(c)$ comes out zero, then the function has a factor $x - c$. It should be noted that c may be integral, rational, or, in fact, any kind of a number.

The process of synthetic division in the foregoing example may be looked upon as a reduction of each term in the polynomial to one of the next lower degree by replacing one of the x's by 2 and combining until the numerical value of the function is obtained.

Thus in $2x^4 - 3x^3 + x^2 - x - 9$, if $x = 2$, we have the first term $2x^4 = 2 \cdot 2x^3 = 4x^3$. Adding to the second term, we have $4x^3 - 3x^3 = x^3$. Letting one $x = 2$, $x^3 = 2x^2$. Adding to the third term, $2x^2 + x^2 = 3x^2$. Substituting 2 for one x, $3x^2 = 6x$. Adding to the fourth term, $6x - x = 5x$. Substituting 2 for x, $5x = 10$. Hence the value of the function for $x = 2$ is $10 - 9 = 1$, which agrees with the result already obtained.

This process is similar to that of converting a distance expressed in yards, feet, and inches into one expressed in inches, by reducing the yards to feet, adding to the number of feet given, reducing this to inches, and adding to the number of inches given.

EXERCISES

1. Find the remainder when $2x^5 - 5x^4 + 4x^3 - 54x^2 - 32x - 30$ is divided by $x - 4$. Do this by direct substitution, as in § 54, and also by the method of synthetic division. Which method is preferable?

2. Find the remainder when $x^4 - 8x^3 + 6$ is divided by $x - 2$. Do this by both methods mentioned in the preceding exercise. Which method is preferable?

3. Find the remainder and the quotient when $3x^4 - 2x^3 + x + 6$ is divided by $x + 3$; by $x - 3$.

4. Find the remainder and the quotient when $4x^3 - 4x^2 - 3x + 2$ is divided by $x - \frac{1}{2}$; by $x + \frac{1}{2}$.

5. Given $f(x) = 8x^3 - 24x^2 - 16x + 40$, find $f(1)$, $f(\frac{1}{2})$, $f(-2)$, $f(8)$, $f(\sqrt{2})$.

6. Find the value of the function $3x^3 - 11x^2 - 18x - 24$, when $x = \frac{2}{3}$; when $x = -\frac{7}{3}$.

7. Show that $x - 2$ and $x + 5$ are factors of $x^4 - 23x^2 + 18x + 40$. What are the other factors? What are the roots of the equation $x^4 - 23x^2 + 18x + 40 = 0$?

8. Show that -3 is a root of the equation $x^3 + 4x^2 - 17x - 60 = 0$. What are the other roots?

9. Show that $x - 1$ is twice a factor, that is, that $(x - 1)^2$ is a factor of $x^4 - x^2 - 2x + 2$, and hence that 1 is a double root of the equation $x^4 - x^2 - 2x + 2 = 0$. What are the other roots?

10. Find the value of k if 2 is a root of the equation $2x^4 - 6x^3 + 4kx + 13 = 0$.

11. Find the values of k and l if -1 and 2 are roots of the equation $3x^4 - 3x^3 - 10x^2 + 2kx - 2l = 0$.

12. For what values of k will 1 be a root of the equation $5x^5 - 4x^3 + 2k^2x^2 + k = 2$?

56. The graphing of functions. We are now in a position to find the graph of a polynomial in the most expeditious manner. We shall symbolize the function $f(x)$ by y and find the values of y corresponding to various values of x.

In plotting, if the table of values consists of numbers which are large or are so distributed that the plot would not be well proportioned if one space on the paper were taken for each unit, a scale should be chosen so that the plot will form a graceful curve.

EXAMPLE

Plot $y = x^3 + 4x^2 - 4$.

Solution. We find by synthetic division the values of y corresponding to various values of x.

The value of y when $x = 0$ is found by direct substitution.

x	-5	-4	-3	-2	-1	0	1	2
y	-29	-4	$+5$	$+4$	-1	-4	$+1$	$+20$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{1} \\ & +1+ & 5+ & 5 & \\ \hline 1+ & 5+ & 5+ & 1 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{2} \\ & +2+ & 12+ & 24 & \\ \hline 1+ & 6+ & 12+ & 20 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{-1} \\ & -1- & 3+ & 3 & \\ \hline 1+ & 3- & 3- & 1 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{-2} \\ & -2- & 4+ & 8 & \\ \hline 1+ & 2- & 4+ & 4 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{-3} \\ & -3- & 3+ & 9 & \\ \hline 1+ & 1- & 3+ & 5 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{-4} \\ & -4+ & 0+ & 0 & \\ \hline 1+ & 0+ & 0- & 4 & \end{array}$$

$$\begin{array}{rrrr|l} 1+ & 4+ & 0- & 4 & \underline{-5} \\ & -5+ & 5- & 25 & \\ \hline 1- & 1+ & 5- & 29 & \end{array}$$

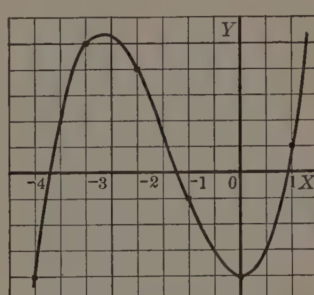
Since the numerical values of y for $x = 2$ and for $x = -5$ are so much larger than the other values of y, and since, as we shall see in the next section, they give us points on the curve which we are not interested in, we shall not include them in making the graph.

In this figure two spaces are taken to represent one unit of x. A single space is taken for a unit of y.

By referring to the graph it is seen that the curve crosses the X axis at about the points $x = .8$, $x = -1.2$, and $x = -3.7$. Since these are approximately the values of x which make y, or $f(x)$, equal to zero, they are approximately the roots of the equation $x^3 + 4x^2 - 4 = 0$.

By performing mentally the algebraic additions in the process of synthetic division, the work in this example may be compressed to the following form:

x			y
0	1 + 4 + 0		-4
1	5	5	1
2	6	12	20
-1	3	-3	-1
-2	2	-4	4
-3	1	-3	5
-4	0	0	-4
-5	-1	5	-29



57. Extent of the table of values. Since the object in plotting a curve is to obtain information regarding the roots of its equation, stretches of the curve beyond all crossings of the X axis are of no interest for the present purpose. Hence it is desirable to know when a table of values has been formed extensive enough to afford a plot

which includes all the real roots. If for all values of x greater than a certain number the curve lies wholly above the axis, there are no real roots greater than that value of x.

By an inspection of the preceding example it appears that if for a given value of x the signs of the partial remainders are all positive, thus affording a positive value of y, any greater value of x will afford only greater positive partial remainders and hence a greater positive value of y. From this point on, the curve must rise as x increases. Hence none of the roots can lie to the right of this point.

Thus *when all the partial remainders are positive, no greater positive value of x need be substituted.*

Similarly, *when the partial remainders alternate in sign, beginning with the coefficient of the highest power of x, no value of x, greater negatively, need be substituted.*

EXERCISES

Graph the following functions. Set each equal to zero, and determine between what consecutive integers the real roots of the resulting equations lie:

1. $x^3 - 3x + 7$.

2. $x^3 - 2x^2 - 10$.

3. $x^3 - 17x + 100$.

4. $2x^3 - 3x^2 - 7x + 5$.

5. $5x^3 - 7x^2 + 3x + 9$.

6. $x^4 + 19x^2 + 11$.

7. $x^4 - 3x^2 - 9x - 31$.

8. $x^4 - 2x^3 - x + 2$.

9. $x^4 - 2x^3 + 3x^2 - 20x - 47$.

10. $6x^4 - 13x^3 + 20x^2 - 37x + 24$.

11. A rectangle whose perimeter is 36 inches is rotated about a line joining the mid-points of two opposite sides, thus generating a cylinder whose volume is 550 cubic inches. Find the dimensions of the rectangle.

NOTE. Take $\pi = \frac{22}{7}$. Use the graph to obtain an approximate result.

12. A rectangle whose perimeter is 33 inches is rotated about a line joining the mid-points of two opposite sides, thus generating a cylinder whose volume is 385 cubic inches. Find the dimensions of the rectangle.

13. Each dimension of a rectangular tank $6 \times 8 \times 10$ feet is to be increased by the same amount, so that the tank will have a capacity of 1000 cubic feet. Estimate from a graph the length each edge is increased.

14. The radii of four spheres are in arithmetical progression, having a common difference of 1 inch. If the largest sphere is equal in volume to the sum of the other three, find their radii.

58. Number of roots. It appeared from the solution of the quadratic equation that every equation of the second degree has two and only two roots. In the preceding exercises it may have been noted that the graph of a function of degree n never crosses the X axis more than n times, that is, none of these equations has more than n roots. Whether the general equation of degree n has any roots at all is a problem which remained unsolved until about a hundred years ago, when it was proved, by methods which we will not reproduce here, that *every rational integral equation of degree n possesses at least one root.* That this root may not be a real number is indicated by problem 6, p. 96, in which the graph does not cross the X axis. But in such a case the theorem would demonstrate the existence of a complex number which satisfies the equation. This Fundamental Theorem of Algebra we shall assume. We can then prove

THEOREM I. *Every equation of degree n in general form has n roots.*

Given the equation $\quad a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$

where the a's are integers, $a_0 \neq 0$, and n is a positive integer. Since the multiplication of an equation by a constant does not affect the roots of the equation in any way (§ 23), we may multiply each member of the equation by the constant $\frac{1}{a_0}$, thus throwing the equation into the p-form. It should be kept in mind that the coefficients $p_1, p_2, \cdots, p_n$, are really nothing else than $\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_n}{a_0}$.

We have, then,

$$x^n + p_1x^{n-1} + \cdots + p_n = 0,$$

an equation which has the same roots as the original equation.

By the Fundamental Theorem of Algebra this equation has at least one root, which we will call x_1. By the Factor Theorem $x - x_1$ must be a factor of the left member. Hence if we write it in factored form, we have

$$(x - x_1)(x^{n-1} + q_1x^{n-2} + \cdots + q_{n-1}) = 0.$$

Now reasoning as before, the expression inside the second parenthesis, set equal to zero, has at least one root, which we will call x_2,

and by the Factor Theorem the expression must have $x - x_2$ as a factor. We may then write the expression within the second parenthesis in factored form. Hence

$$(x - x_1)(x - x_2)(x^{n-2} + s_1x^{n-3} + \cdots + s_{n-2}) = 0.$$

We may continue this process until the last factor is of the first degree, which, set equal to zero, will have a root which we may call x_n. We have then the second equation with its left member expressed as the product of n linear factors,

$$(x - x_1)(x - x_2)\cdots(x - x_n) = 0.$$

The roots of this equation are $x_1, x_2, \cdots, x_n$, which are evidently n in number.

Not all of these roots need be distinct. If two of the roots, say x_1 and x_2, are equal to each other, $f(x)$ will have $(x - x_1)^2$ as a factor. We say that x_1 is then a double root of $f(x) = 0$. If r roots are equal to each other, $f(x)$ will have r equal linear factors, and $f(x) = 0$ will have an r-fold root, or a multiple root of order r. Multiple roots may be regarded as limiting cases of roots which have been approaching each other and have finally become equal.

It should be particularly noted that certain of these roots $x_1, x_2, \cdots, x_n$, may be complex numbers, so that these linear factors are not necessarily of the simple type considered in § 1. It was stated in that section that it is sometimes desirable to find factors whose coefficients are not integers, rational fractions, or even real numbers. With this understanding we may state as a result of our theorem that the general polynomial may be expressed as the product of linear factors.

Assumption. *If $x - a$, $x - b$, $x - c, \cdots, x - k$, are each factors of a polynomial, then their product is a factor of the polynomial.*

Theorem II. *The general equation of the nth degree, $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, has no more than n roots.*

For if $x_1, x_2, \cdots, x_n, x_{n+1}$, are each roots of the given equation, then, by the Factor Theorem, $x - x_1, x - x_2, \cdots, x - x_n, x - x_{n+1}$, are each factors of the left member. Hence, by the preceding assumption, their product is contained in the left member, which would therefore have to be at least of degree $n + 1$, which is contrary to the hypothesis.

THEOREM III. *If the equation* $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ *is satisfied by more than n distinct values of x, all of its coefficients must vanish.*

The distinction between this theorem and the preceding lies in the hypothesis. There it was assumed that $a_0 \neq 0$ (§ 53), and we found the number of values of x which could satisfy the equation. Here our hypothesis states that the equation is satisfied by more than n values of x, and we propose to determine what happens to the coefficients.

Illustration. This theorem finds no application when we are dealing with equations with numerical coefficients, for in that case if any term has a zero coefficient it simply drops out. But we might have the quadratic equation

$$(a-2)x^2 + (a^2-4)x + a^2 - 3a + 2 = 0,$$

which we had found in some way was satisfied by the three numbers

$$x = 3, \quad x = 2, \quad \text{and} \quad x = 1.$$

The theorem tells us that a must have such a value that all of the coefficients vanish; that is, a must equal 2.

Proof. Suppose that not all of the coefficients vanish. Then the degree of the polynomial will be n or perhaps less than n. Therefore, by the preceding theorem, it cannot have more than n roots. But by the hypothesis it must have $n+1$ roots. Therefore the assumption that not all of the coefficients vanish is false.

COROLLARY. *If two polynomials in x of degree n are equal to each other for more than n values of x, the coefficients of like powers of x are equal to each other.*

We have given

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

for more than n values of x. Transposing, we have

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0.$$

By Theorem III,

$$\begin{aligned} a_0 - b_0 &= 0, \quad \text{or} \quad a_0 = b_0, \\ a_1 - b_1 &= 0, \quad \text{or} \quad a_1 = b_1, \\ &\cdots\cdots\cdots \\ a_n - b_n &= 0, \quad \text{or} \quad a_n = b_n. \end{aligned}$$

EXERCISES

1. If the equation $a^2(x^2+x+1)+3a(x^2+2)-9(x-1)=0$ is satisfied by $x=2$, $x=3$, and $x=5$, find the value of a.

2. If the polynomial $(3a+b-3c)x^2+(a+b+c)x-(2a+b-c)$ vanishes when x has three different values, what must be the values of a, b, and c?

3. If the equation $ax^3+b^2(x^2+x)+b(x^2+2x)+x-1+c^2=0$ is satisfied for four different values of x, what are the values of a, b, and c?

4. If $19x+1=A(3x-1)+B(5x+2)$ for all values of x, find the values of A and B.

5. If $x^2-2=Ax(x-2)+Bx(x-1)+C(x-1)(x-2)$ for all values of x, find the values of A, B, and C.

59. Complex roots. In the exercises on page 96 it was noted that the graphs of some of the functions cross the X axis fewer times than the degree of the corresponding equation; for instance, the graph for the second exercise crosses but once. Since crossing the axis indicates a real root, and since every equation must have n roots, real or complex, we can tell from the graph how many complex roots an equation has.

THEOREM. *If a rational integral equation with real coefficients has the complex number $c+id$ for one of its roots, it must also have the number $c-id$ for a root.*

Given the equation

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0,$$

where the a's are real numbers, and given that $c+id$ is a root of this equation, it is required to prove that $c-id$ is a root.

To say that $c+id$ is a root of the given equation means that if $c+id$ is substituted for x, the equation is satisfied; that is,

$$f(c+id)=a_0(c+id)^n+a_1(c+id)^{n-1}+\cdots+a_n=0.$$

Now if we expand each of the powers of $c+id$ by the Binomial Theorem,* we obtain an expression some of whose terms contain no i, while others contain i to various powers, from i to i^n. But, by § 39, any power of i reduces to 1, -1, i or $-i$, so that finally each term

* Let the student write out these expansions to several terms.

in the expansion will either contain no i at all or it will contain i to the first power. Now let us group together all terms of the expansion which do not contain i. Denote this group of terms by P. Then group together all terms of the expansion containing i and denote the expression representing the complete coefficient of i by Q. Then we may write

$$f(c+id)=P+iQ=0.$$

Hence, by § 42, we must have $P=0$ and $Q=0$.

Now we have to show that $c-id$ is a root of the given equation; that is, we must show that $f(c-id)=0$. Let us form the expression $f(c-id)$. This may be obtained from the expression we have found for $f(c+id)$ by changing i to $-i$; that is, we have

$$f(c-id)=P-iQ,$$

where P and Q represent the same expressions as before.

But we proved above that these expressions P and Q must each equal 0. Hence

$$f(c-id)=0;$$

that is, $c-id$ is a root of the given equation.

Illustration. Consider the equation

$$f(x)=x^3+px+q=0. \tag{1}$$

Let $c+id$ be a root of (1); we will prove that $c-id$ is also a root. Since $c+id$ is a root, we have

$$f(c+id)=(c+id)^3+p(c+id)+q=0.$$

Expanding the first term by the Binomial Theorem,

$$\begin{aligned} f(c+id)&=c^3+3c^2id+3c(id)^2+(id)^3+pc+pid+q\\ &=c^3-3cd^2+pc+q+i(3c^2d-d^3+pd)\\ &=P+iQ=0,\end{aligned}$$

where $P=c^3-3cd^2+pc+q$; $Q=3c^2d-d^3+pd$.

By § 42, $P=0$, $Q=0$.

Now
$$\begin{aligned} f(c-id)&=(c-id)^3+p(c-id)+q\\ &=c^3-3cd^2+pc+q-i(3c^2d-d^3+pd)\\ &=P-iQ.\end{aligned}$$

But we have shown that $P=0$ and $Q=0$. Hence $f(c-id)=0$ and $c-id$ is a root of (1).

COROLLARY. *Every equation of odd degree with real coefficients has at least one real root.*

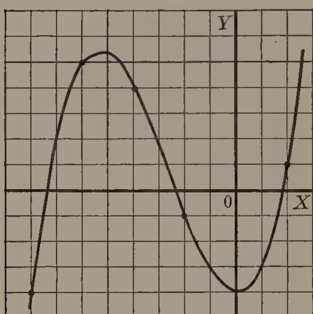
The roots cannot all be complex, else the degree of the equation would be even by the preceding theorem.

60. Multiple roots. When we plot the equations

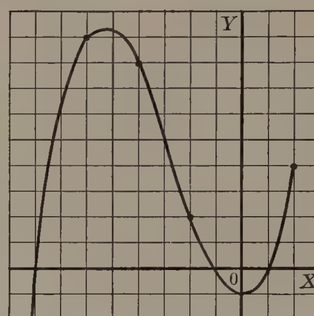
$$y = x^3 + 4x^2 - 4, \qquad (1) \qquad\qquad y = x^3 + 4x^2 - 1, \qquad (2)$$

$$y = x^3 + 4x^2, \qquad (3) \qquad\qquad y = x^3 + 4x^2 + 1, \qquad (4)$$

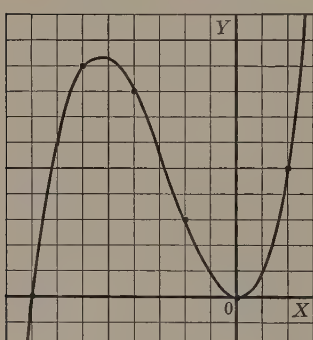
we see that corresponding to the increase of the constant term there is a corresponding elevation of the curve with respect to the X axis. In every case the curve is the same, but the corresponding values



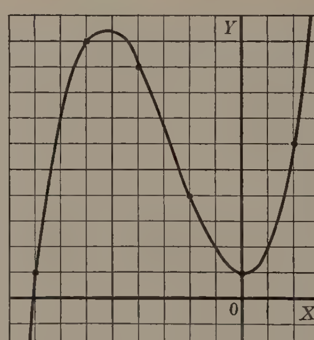
$y = x^3 + 4x^2 - 4$, (1)



$y = x^3 + 4x^2 - 1$, (2)



$y = x^3 + 4x^2$, (3)



$y = x^3 + 4x^2 + 1$, (4)

of y are different. In (1) and (2) the curve crosses the X axis three times, in (3) it touches the X axis, and in (4) we have only one crossing. Hence equations (1) and (2) each have three real roots; (3) also has three real roots, where one root is counted twice; and (4) has only one real root. As the curve is raised, an inspection of the right elbow of the figure shows that two of the roots approach nearer to

each other, and finally coincide in (3), forming the double root. As it is further raised, this elbow fails to intersect the X axis, and the pair of roots has ceased to be real. But since a cubic equation always has three roots, a pair of roots must have become complex. Consequently, a double root is, in a certain sense, a limiting case between two roots which are real and distinct, and a pair of complex roots. Thus we have the

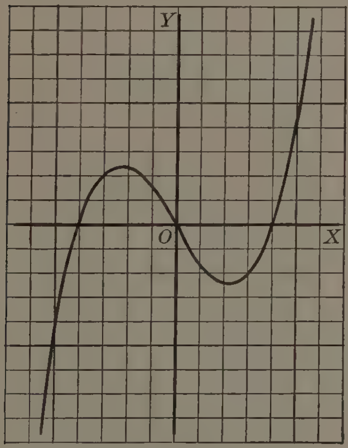
PRINCIPLE. *Corresponding to every elbow of the curve that does not intersect the X axis there is a pair of complex roots of the equation.*

The converse is not always true. It is not always possible to find as many elbows of the curve which do not meet the X axis as there are pairs of complex roots.

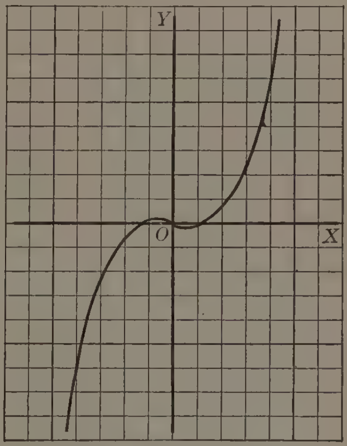
Consider the equations

$$y = x^3 - 16x, \quad (5) \qquad y = x^3 - x, \quad (6) \qquad y = x^3. \quad (7)$$

The equation $x^3 - 16x = 0$ has the roots -4, 0, and $+4$. The graph of (5) consequently cuts the X axis at the points -4, 0, and



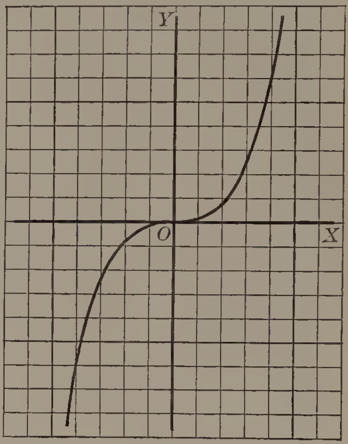
$y = x^3 - 16x$



$y = x^3 - x$

$+4$, and has two elbows. The equation $x^3 - x = 0$ has the roots -1, 0, and $+1$, and the graph of (6) also has two elbows. But since the points where it crosses the X axis are closer together than in the case of (5), the elbows are less prominent. Finally, in equation $x^3 = 0$ the roots are equal to each other, and the corresponding curve has no elbows at all, but crosses the axis only at the origin.

The forms of the graphs of the equations here considered indicate that as three real roots of an equation become more nearly equal, two elbows of the corresponding graph become less and less prominent until, when the roots are equal, the elbows have faded away entirely. Hence the graph of an equation crosses the X axis only once for a three-fold root.



$y = x^3$

It should be observed that the foregoing curves belong to the system whose equation is $x^3 - ax = y$, and that the values of a are 16, 1, and 0.

Similarly, we could show that if an equation has an n-fold root, the corresponding graph crosses the X axis if n is odd, and touches it if n is even.

61. Binomial quadratic surd roots. By a **binomial quadratic surd** is meant a number of the form $\sqrt{a} + \sqrt{b}$, where a and b are positive rational numbers but are not both perfect squares.

THEOREM I. *If a binomial quadratic surd of the form $a + \sqrt{b}$ is equal to zero, then $a = 0$ and $b = 0$.*

If $a + \sqrt{b} = 0$ and either $a = 0$ or $b = 0$, clearly both must equal zero.

Suppose, however, that neither a nor b equals zero. Then, transposing, we have $a = -\sqrt{b}$, and a rational number is equal to an irrational number, which is impossible.

Hence the only alternative is that both a and b equal zero.

Following the terminology used in the definition of conjugate complex numbers (§ 46), $a + \sqrt{b}$ and $a - \sqrt{b}$ are called **conjugate binomial surds.**

THEOREM II. *If a given binomial surd $a + \sqrt{b}$ is a root of an equation with rational coefficients, then its conjugate is also a root of the same equation.*

The proof of this theorem may be made analogously to the proof of the theorem on page 100.

62. Relations between roots and coefficients. If the equation in p-form,

$$x^n + p_1x^{n-1} + \cdots + p_n = 0, \qquad (P)$$

has for its roots the numbers $x_1, x_2, \cdots, x_n$, we have seen in § 58 that it may be written in factored form as follows:

$$(x - x_1)(x - x_2)\cdots(x - x_n) = 0.$$

If we multiply out the left member of this equation, collect like powers of x, and compare the various coefficients obtained with the coefficients in (P), we shall find certain relations between the roots and the coefficients.

Illustration. Let $n = 4$; that is, let the equation be

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0, \qquad (P)$$

or, in factored form, $(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0.$

Multiplying out, we have

$$\begin{aligned}(x - x_1)(x - x_2)(x - x_3)(x - x_4) &= x^4 - (x_1 + x_2 + x_3 + x_4)x^3 \\ &+ (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2 \\ &- (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x + x_1x_2x_3x_4 = 0.\end{aligned}$$

Equating coefficients of like powers of x in this equation and in (P), we have

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 &&= -p_1, &(1)\\ &x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &&= p_2, &(2)\\ &x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 &&= -p_3, &(3)\\ &x_1x_2x_3x_4 &&= p_4. &(4)\end{aligned}$$

This result suggests the following general theorem for the equation of degree n in p-form:

THEOREM. (a) *The sum of the roots equals the coefficient of the second term with its sign changed.*

(b) *The sum of the products of the roots taken two at a time equals the coefficient of the third term.*

(c) *The sum of the products of the roots taken three at a time equals the coefficient of the fourth term with its sign changed.*

(d) *The product of the roots equals the constant term, with its sign changed if n is odd.*

We will now prove parts (a) and (d) for an equation of any degree. We will first show that if (a) and (d) are true for an equation of any degree k, they will also be true for one of degree $k + 1$.

Assuming, then, that (a) and (d) are valid for any equation of degree k, we may write the equation in the form

$$x^k - (x_1 + x_2 + \cdots + x_k)x^{k-1} + \cdots \pm (x_1 x_2 \cdots x_k) = 0. \tag{5}$$

Let us now multiply each member of this equation by $x - x_{k+1}$, thus obtaining an equation of degree $k+1$ with the roots x_1, $x_2, \cdots, x_k, x_{k+1}$.

$$\begin{array}{l} x^k - (x_1 + x_2 + \cdots + x_k)x^{k-1} + \cdots \pm (x_1 x_2 \cdots x_k) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x - x_{k+1} \\ \hline x^{k+1} - (x_1 + x_2 + \cdots + x_k)x^k + \cdots \pm (x_1 x_2 \cdots x_k)x \\ \qquad\qquad\qquad\qquad - x_{k+1}x^k + \cdots\cdots\cdots \mp x_1 x_2 \cdots x_k x_{k+1} \\ \hline x^{k+1} - (x_1 + x_2 + \cdots + x_k + x_{k+1})x^k + \cdots \mp x_1 x_2 \cdots x_k x_{k+1} = 0. \end{array} \tag{6}$$

It is observed that the coefficient of the second term and the constant term of (6) are of the form required by (a) and (d). We have shown, then, that if we can write an equation of degree k, with roots $x_1, x_2, \cdots, x_k$, in form (5), that is, if (a) and (d) are true for an equation of degree k, then (a) and (d) must also be true for an equation of degree $k+1$.

But we know from (1)–(4) that (a) and (d) are true for an equation of degree $k = 4$. Hence the demonstration shows that (a) and (d) are true for an equation of degree $k + 1 = 5$. But if they are true for an equation of degree 5, they must also hold for one of degree 6, and so on, for an equation of any higher degree.*

In applying the above theorem the coefficient of a missing term is taken as 0.

This method of demonstration by which a fact which is known to be true for a certain value of k is proved to be true for the value $k+1$, and hence for all succeeding integral values, is called the **Method of Complete Induction.**

Corollary I. *If x_1 is a root of an equation in p-form with integral coefficients, it is a factor of its constant term.*

This follows immediately from (d).

Corollary II. *If the rational number $\frac{a}{b}$ is a root of the equation in a-form with integral coefficients, $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, then b is a factor of a_0, and a is a factor of a_n.*

* For a proof of the complete theorem, see Hawkes, Advanced Algebra, p. 177.

Illustration. Suppose that the numbers $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$ are roots of a cubic equation. Then the equation must be expressible in factored form as follows:

$$(x - \tfrac{1}{2})(x - \tfrac{2}{3})(x - \tfrac{3}{5}) = 0.$$

Each of these factors may be written with a common denominator, and the equation becomes

$$\left(\frac{2x - 1}{2}\right)\left(\frac{3x - 2}{3}\right)\left(\frac{5x - 3}{5}\right) = 0,$$

or, after multiplying through by 30, in order to make the coefficients integral,

$$(2x - 1)(3x - 2)(5x - 3) = 0.$$

Finally, multiplying out,

$$30x^3 - 53x^2 + 31x - 6 = 0. \tag{1}$$

From this form it appears that the coefficient of x^3 in (1) is exactly the product of the denominators of the fractional roots, and that the constant term 6 is the product of the numerators of the roots, except for sign. Furthermore, if equation (1) had been given and we wished to determine its fractional roots, we would only need to consider fractions whose denominators are factors of 30 and whose numerators are factors of 6.

Proof. Let the equation be denoted by $f(x) = 0$. If $\frac{a}{b}$ is a rational root, then $x - \frac{a}{b}$ must be a factor of $f(x)$. Hence the equation may be written in the form

$$\left(x - \frac{a}{b}\right)Q(x) = 0, \quad \text{or} \quad \left(\frac{bx - a}{b}\right)Q(x) = 0, \tag{2}$$

where $Q(x)$ has integral coefficients. Multiplying through by b, we obtain $(bx - a)Q(x) = 0$, from which it appears that b is a factor of the coefficient of the highest power of x, and a is a factor of the constant term of the expression in the left member of the equation $f(x) = 0$.

Illustration. Let $f(x) = 0$ be (1) above, and let $\frac{a}{b} = \frac{1}{2}$. Then dividing the left member of (1) by $x - \frac{1}{2}$, we find that $Q(x) = 30x^2 - 38x + 12$. Hence $(bx - a)Q(x) = 0$ becomes in this case $(2x - 1)(15x^2 - 19x + 6) = 0$. It is observed that in dividing the polynomial $f(x)$ by $x - \frac{a}{b}$, each coefficient of the quotient $Q(x)$ contains b as a factor, which may be divided out when the expression is set equal to zero.

63. Formation of equations with known roots. If we know all of the roots of an equation, we may form the equation in either of two ways: the first method uses the principle of the Factor Theorem; the second employs the relations between the roots and the coefficients derived in the preceding section.

First method. *If $x_1, x_2, \cdots, x_n$, are the roots, multiply together the factors $x - x_1, x - x_2, \cdots, x - x_n$, and set the product equal to* 0.

Second method. *From the roots find the coefficients, using the relations of* § 62.

If the equation and all but one of its roots are known, that root can be found most readily by the solution of the linear equation obtained by setting the sum of the roots equal to the coefficient of the second term with its sign changed.

If all but two of the roots are known, the unknown roots may be found by the solution of a pair of simultaneous equations formed by using the coefficient of the second term and the last term.

In using the second method the equation must always be in p-form.

EXAMPLES

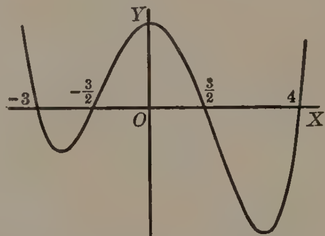
1. Find the equation having for roots the numbers $4, -3, \pm \frac{3}{2}$. Draw the graph of the function forming the left member of the equation.

First solution. Applying the Factor Theorem, we may write the equation in the form $(x-4)(x+3)(x-\frac{3}{2})(x+\frac{3}{2}) = 0$,

or $\quad (x-4)(x+3)(2x-3)(2x+3)=0.$

By multiplication we find

$$4x^4 - 4x^3 - 57x^2 + 9x + 108 = 0. \quad (1)$$



The graph of the function

$$y = 4x^4 - 4x^3 - 57x^2 + 9x + 108$$

crosses the X axis at the points where $x = 4$, $\frac{3}{2}, -\frac{3}{2}, -3$. These numbers are the roots of equation (1).

Second solution. We may find the equation in p-form by applying the results of § 62. We have

$$p_1 = -(4 - 3 + \tfrac{3}{2} - \tfrac{3}{2}) = -1.$$

$$p_2 = 4(-3) + 4(\tfrac{3}{2}) + 4(-\tfrac{3}{2}) + (-3)(\tfrac{3}{2}) + (-3)(-\tfrac{3}{2}) + (\tfrac{3}{2})(-\tfrac{3}{2})$$
$$= -12 - \tfrac{9}{4} = -\tfrac{57}{4}.$$

$$p_3 = -\{4(-3)(\tfrac{3}{2}) + 4(-3)(-\tfrac{3}{2}) + 4(\tfrac{3}{2})(-\tfrac{3}{2}) + (-3)(\tfrac{3}{2})(-\tfrac{3}{2})\}$$
$$= -\{-\tfrac{36}{4} + \tfrac{27}{4}\} = \tfrac{9}{4}.$$

$$p_4 = 4(-3)(\tfrac{3}{2})(-\tfrac{3}{2}) = \tfrac{108}{4}.$$

Hence the equation is $\quad x^4 - x^3 - \frac{57}{4}x^2 + \frac{9}{4}x + \frac{108}{4} = 0.$

Writing this in the a-form, we have

$$4x^4 - 4x^3 - 57x^2 + 9x + 108 = 0.$$

2. Solve the equation

$$2x^4 + 7x^3 + 14x^2 + 11x - 10 = 0,$$

given that one root is $-1 + \sqrt{2}\,i$.

Solution. Writing the equation in p-form,

$$x^4 + \tfrac{7}{2}x^3 + 7x^2 + \tfrac{11}{2}x - 5 = 0.$$

By the theorem of § 59 the equation has a root $-1-\sqrt{2}\,i$. Since the equation is of the fourth degree, it has four roots. Let the unknown roots be denoted by r and s. The sum of the two known roots, $-1+\sqrt{2}\,i$ and $-1-\sqrt{2}\,i$, is -2 and their product is 5. Then, by § 62, we have

the sum of the roots, $\qquad r + s - 2 = -\tfrac{7}{2}$,

the product of the roots, $\qquad 5rs = -5$;

or $\qquad r + s = -\tfrac{3}{2} \qquad (1)$

and $\qquad rs = -1, \qquad (2)$

a set of two equations, to be solved for r and s.

From (2), $s = -\dfrac{1}{r}$. Substituting this in (1),

$$r - \frac{1}{r} = -\frac{3}{2},$$

$$2r^2 + 3r - 2 = 0,$$

$$(r+2)(2r-1) = 0, \quad r = -2 \text{ or } \tfrac{1}{2};$$

and substituting these values of r in (2) we obtain

$$s = \tfrac{1}{2} \text{ or } -2.$$

The other two roots are, then, -2 and $\tfrac{1}{2}$.

Therefore the four roots of the original equation are

$$-2,\ \tfrac{1}{2},\ -1 \pm \sqrt{2}\,i.$$

Check. The equation whose roots are $-1 \pm \sqrt{2}\,i$ is $x^2 + 2x + 5 = 0$. The equation whose roots are -2, $\tfrac{1}{2}$ is $2x^2 + 3x - 2 = 0$.

$$(x^2 + 2x + 5)(2x^2 + 3x - 2) = 2x^4 + 7x^3 + 14x^2 + 11x - 10 = 0,$$

which is the given equation.

EXERCISES

Find the equations having the following roots. One method of § 63 may be used to check the other. Draw roughly, without making a table of values, the graph of the function forming the left member each of the equations in exercises 1–15, noting that each real the equation represents a point where the graph crosses or the X axis. The graph crosses the X axis if the real root is root or multiple root of odd order, and touches it if the root

is a multiple root of even order (see § 60). If the coefficient of the highest power of x is positive, the value of y is positive for large positive values of x.

1. $1, -2$.

2. $2, 3, -4$.

3. $2, 1, 1, 0$.

4. $\pm 3, \pm 6$.

5. $0, 0, 0, 2$.

6. $-1, -1, 0, 0, 0$.

7. $0, 0, 0, 0, -3$.

8. $1, 2, \pm \frac{1}{2}$.

9. $\frac{2}{3}, \frac{1}{3}, -1$.

10. $2, \frac{1}{2}, 3, \frac{1}{3}$.

11. $\pm\sqrt{2}, 0, 1$.

12. $\pm\sqrt{3}, \pm\sqrt{5}$.

13. $1 \pm \sqrt{2}, -2$.

14. $2 \pm \sqrt{3}, -2 \pm \sqrt{3}$.

15. $0, 0, 0, -1 \pm \sqrt{5}$.

16. $1, \pm i\sqrt{2}$.

17. $-1, 1 \pm i$.

18. $2 \pm i, -2 \pm i$.

19. $0, 0, \pm i, \pm 2i$.

20. $1, \dfrac{-1 \pm i\sqrt{3}}{2}$.

21. $\dfrac{1 \pm i\sqrt{3}}{2}, \dfrac{-1 \pm i\sqrt{3}}{2}$.

22. Form an equation of the second degree one of whose roots is $\dfrac{1 + 2\sqrt{-5}}{3}$.

23. Form an equation of the third degree two of whose roots are $-3, \dfrac{-1 - i\sqrt{23}}{2}$.

24. Form an equation of the fourth degree two of whose roots are $i, \dfrac{1 + i\sqrt{3}}{2}$.

25. Form an equation of lowest possible degree with real coefficients having the two numbers $\pm 1 + i$ for roots.

26. Solve the equation $x^3 - x^2 - 22x + 40 = 0$, given that one root is double another.

27. Solve the equation $x^4 - 8x^3 + 11x^2 + 32x - 60 = 0$, given that the sum of two of the roots is 0.

28. Solve the equation $x^3 - 12x^2 + 23x + 36 = 0$, given that the roots are in arithmetical progression.

29. Solve the equation $4x^3 + 12x^2 - 67x + 30 = 0$, given that the sum of two of the roots is 3.

30. Solve the equation $x^3 - 14x^2 - 31x - 16 = 0$, given that two of the roots are equal.

31. Solve the equation $4x^3 + 9x^2 - 30x - 8 = 0$, given that one root is the reciprocal of another.

32. Solve the equation $x^4 - 4x^3 + 5x^2 + 8x - 14 = 0$, given that one root is $2 - i\sqrt{3}$.

33. Solve the equation $(x-4)^2 + 2(x-4) = \frac{2}{x} - 1$, given that one root is $2 + \sqrt{3}$.

34. Solve the equation $x^4 + 12x^3 + 78x^2 + 252x + 272 = 0$, given that $-3 + 5i$ is a root.

35. Solve the equation $x^5 - 2x^4 - x^3 + 2x^2 + 10x = 0$, given that $2 - i$ is a root.

36. Solve the equation $x^5 - 12x^3 + 46x^2 - 85x + 50 = 0$, given that two of the roots are $1, 1 + 2i$.

37. Solve the equation $2x^3 + 13x^2 - 26x - 16 = 0$, given that the roots are in geometrical progression.

38. (a) Solve the equation $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$, given that the sum of two of the roots is equal to the sum of two others, and that one root is the negative of another.

(b) Determine another equation of the fourth degree having, like the above equation, the sum of its roots equal to -6, their product equal to -8, the sum of two of its roots equal to the sum of two others, and one root the negative of another. What are the roots of this equation?

39. What must be the value of k if the sum of three of the roots of the equation $x^4 - 3x^3 = kx - 9$ is 0?

40. Show that an equation $Ax^4 + Bx + C = 0$, where the coefficients are real, cannot have four real roots unless B and C are both zero, in which case all the roots are zero.

64. Detection of rational roots. In the following sections we shall apply ourselves to the problem of finding the numerical values of the roots of a rational integral equation with integral coefficients.

The simplest type of root, and the one which is easiest to find, is the integer. By Corollary I, § 62, any root of an equation in p-form with integral coefficients is a factor of the constant term. Hence no integers other than such factors need be tried in any particular case of this kind.

For example, if the constant term in an equation in p-form with integral coefficients is 3, the only possible integral roots of the

equation are ± 1 and ± 3. If it is found by synthetic division that none of these is a root of the equation, we must conclude that the equation has no integral roots.

The existence of any rational root may be determined by the use of Corollary II, § 62. For example, consider the equation

$$6x^3 - x^2 - 3x - 20 = 0.$$

First obtain the table of values for the function in the left member as if to plot it.

x	0	1	2	-1
y	-20	-18	$+18$	-24

From this table it appears that there is a real root between $x = 1$ and $x = 2$, since y is negative for $x = 1$, and positive for $x = 2$. Hence we seek fractions of the form $\frac{a}{b}$, of value between 1 and 2, such that a is a factor of 20, and b is a factor of 6. The factors of 20 are ± 1, ± 2, ± 4, ± 5, ± 10, ± 20. The factors of 6 are ± 1, ± 2, ± 3, ± 6. The only fractions which satisfy the conditions of the problem are $\frac{4}{3}$ and $\frac{5}{3}$. These values we try by synthetic division.

$$\begin{array}{l} 6 - 1 - 3 - 20 \,\underline{|\frac{4}{3}} \\ \underline{\quad + 8 + \frac{28}{3} + \frac{76}{9}} \\ 6 + 7 + \frac{19}{3} - \frac{104}{9} \end{array} \qquad \begin{array}{l} 6 - 1 - 3 - 20 \,\underline{|\frac{5}{3}} \\ \underline{\quad + 10 + 15 + 20} \\ 6 + 9 + 12 + 0 \end{array}$$

Since the fraction $\frac{5}{3}$ gives the remainder 0, we have shown that this is a root of the equation. The remaining roots of the equation may now be found by solving the quadratic equation

$$6x^2 + 9x + 12 = 0,$$

formed by setting the quotient of the division equal to 0. They are

$$\frac{-3 \pm \sqrt{-23}}{4}.$$

When the coefficient of the highest power of x in an equation is equal to unity, the denominator of any fractional root must have the value 1, since this is the only integer which is a factor of that coefficient. This is equivalent to the statement that if an equation in p-form with integral coefficients has any rational roots, they must be integral. From this point of view Corollary I, § 62, is a particular case of Corollary II.

EXAMPLE

Solve the equation

$$2x^4 - x^3 - 5x^2 + 7x - 6 = 0.$$

Solution. First obtain a table of values for the function.

x	-2	-1	0	1	2
y	0	-15	-6	-3	12

$$\begin{array}{l} 2-1-5+7-\ 6 \underline{|2} \\ \underline{\ \ +4+6+2+18} \\ 2+3+1+9+12 \end{array} \qquad \begin{array}{l} 2-1-\ 5+\ 7-6 \underline{|-2} \\ \underline{\ \ -4+10-10+6} \\ 2-5+\ 5-\ 3+0 \end{array}$$

The values of the function when $x = 0$, 1, and -1 are obtained by inspection on substituting these values of x in the function. The values of the function when $x = 2$ and -2 are obtained by synthetic division. When we try 2 by synthetic division we notice that the numbers in the last line are all positive. Hence the equation cannot have a root larger than 2 (§ 57), and it is unnecessary to try larger numbers.

When we try -2 by synthetic division we notice that the numbers in the last line alternate in sign. Hence the equation cannot have a root smaller than -2, and it is unnecessary to try smaller numbers. This division also shows that -2 is a root.

The remaining roots of the original equation must be the roots of the reduced equation formed by setting the quotient of the division by $x + 2$ equal to zero; namely,

$$2x^3 - 5x^2 + 5x - 3 = 0.$$

This equation must have a root between 1 and 2, since we see from the table that the original equation has a root between 1 and 2, as the function is negative when $x = 1$ and positive when $x = 2$. If this root between 1 and 2 is rational, it must be a fraction whose numerator is a factor of 3 and whose denominator is a factor of 2. The only possibility is $\frac{3}{2}$. We try this value by synthetic division.

$$\begin{array}{l} 2-5+5-3 \underline{|\tfrac{3}{2}} \\ \underline{\ \ +3-3+3} \\ 2-2+2+0 \end{array}$$

Hence $\frac{3}{2}$ is a root. The quotient of this division, set equal to zero, is a quadratic equation which gives the remaining roots

$$2x^2 - 2x + 2 = 0,$$

$$x^2 - x + 1 = 0,$$

$$x = \frac{1 \pm \sqrt{-3}}{2}.$$

Therefore the roots of the original equation are $-2, \frac{3}{2}, \frac{1 \pm \sqrt{-3}}{2}$.

EXERCISES

Solve the following equations:

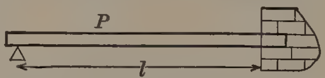
1. $4x^3 - 8x^2 - x + 2 = 0.$
2. $3x^3 + 13x^2 + 11x - 14 = 0.$
3. $4x^3 + 3x^2 - 20x - 15 = 0.$
4. $27x^3 + 63x^2 + 30x - 8 = 0.$
5. $2x^3 - 15x^2 + 46x - 42 = 0.$
6. $16x^3 - 5x - 3 = 0.$
7. $2x^4 - 7x^3 - x^2 + 21x - 15 = 0.$
8. $x^4 + 9x^3 + 5x^2 - 23x + 8 = 0.$
9. $2x^4 + 2x^3 - x^2 + 1 = 0.$
10. $10x^4 - 21x^3 - 21x - 10 = 0.$
11. $4x^4 - 23x^2 + 15x + 9 = 0.$
12. $9x^4 + 15x^3 - 143x^2 + 41x + 30 = 0.$
13. $x^5 - 10x^2 + 15x - 6 = 0.$
14. $12x^5 + 4x^4 - 17x^2 + 7x - 2 = 0.$
15. $x^5 - x^4 + x^3 - x^2 + x - 1 = 0.$

16. Show that the equation $3x^3 + x - 1 = 0$ has no rational roots. Determine how many real roots the equation has.

17. Find a rational root of the equation $2x^4 - 13x^3 + 16x^2 - 9x + 20 = 0$. Show that the equation has only one rational root.

18. Show that the equation $3x^4 - 2x^3 - 21x^2 - 4x + 11 = 0$ has four real irrational roots.

19. A beam of span l is fixed at one end and rests on a support at the other end. The distance of the load P from the supported end being kl, where k is positive and less than 1, the position of P which gives the maximum positive moment is given by the equation $2k^3 - 3k + 1 = 0$. Show that for the maximum positive moment the load is at a distance $.366\,l$ from the supported end.



SUGGESTION. In the three following exercises take $\pi = \frac{22}{7}$.

20. A rectangle whose perimeter is 34 inches is rotated about a line joining the mid-points of two opposite sides. If the volume of the cylinder generated is 550 cubic inches, find the lengths of the sides of the rectangle.

21. The altitude of a cone exceeds the radius of the base by 2 inches and its volume is 462 cubic inches. Find the altitude of the cone and the radius of the base.

22. The sum of the radius of the base and the altitude of a right circular cone is 10 inches and its volume is 66 cubic inches. Find the altitude and the radius of the base.

23. A spherical shell an inch thick, whose outer diameter is 12 inches, is equal in volume to the sum of two spheres whose radii differ by 1 inch. Find the radii of the spheres.

24. It is desired to double the capacity of a tank $6 \times 8 \times 10$ feet by making equal elongations of its dimensions. Find the elongations.

25. The volume of a rectangular parallelepiped is 60 cubic feet. Its total surface is 94 square feet and the total length of its edges is 48 feet. Form the equation whose roots are the dimensions of the parallelepiped and find these dimensions.

65. Multiplication of the roots of an equation by a constant. Suppose we have given the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0. \tag{A}$$

Call its roots $x_1, x_2, \cdots, x_n$. It is required to find an equation whose roots are equal to these numbers each multiplied by k; that is, we seek the equation whose roots are

$$kx_1, kx_2, \cdots, kx_n. \tag{1}$$

Consider the equation

$$f\left(\frac{x'}{k}\right) = a_0\left(\frac{x'}{k}\right)^n + a_1\left(\frac{x'}{k}\right)^{n-1} + \cdots + a_n = 0, \tag{2}$$

where x' is the variable. We shall show that this equation is satisfied by the numbers (1). Replacing x' by any one of the numbers (1), say by kx_1, the polynomial in the left member becomes

$$f\left(\frac{kx_1}{k}\right) = f(x_1).$$

But $f(x_1) = 0$, since x_1 is a root of the equation $f(x) = 0$. Hence equation (2) is satisfied by the numbers (1).

If we remove the parentheses in (2), multiply through by k^n, and drop the primes, we have

$$a_0x^n + a_1kx^{n-1} + a_2k^2x^{n-2} + \cdots + k^na_n = 0, \tag{3}$$

which is the equation sought, having roots each k times the roots of equation (A).* We may express the result in the following

RULE. *To find an equation whose roots are equal to the roots of (A) each multiplied by the constant k, multiply the terms of (A), beginning with the second, by $k, k^2, \cdots, k^n$ respectively.*

* It should be kept in mind that a given equation has the same roots whether the variable is called x or x'.

The special case of this rule for the value $k = -1$ may be expressed as follows:

To find an equation in general form whose roots are the negatives of the roots of a given equation, change the signs of alternate terms.

Care must be taken in applying the above principles to supply missing terms by zeros.

66. Descartes's rule of signs. A pair of successive like signs in a polynomial is called a **continuation** of sign. A pair of successive unlike signs is called a **change** of sign.

In the polynomial $\quad f(x) = 2x^4 - 3x^3 + 2x^2 + 2x - 3 \qquad (1)$
are one continuation of sign and three changes of sign. This may be seen more clearly by writing merely the signs, $+ - + + -$.

Let us now determine the effect on the number of changes of sign in a polynomial if it is multiplied by a factor of the form $x - a$ where a is positive; that is, where the number of positive roots of the equation $f(x) = 0$ is increased by one.

Illustration. Let us multiply (1) by $x - 2$. We have then

$$\begin{array}{l} 2x^4 - 3x^3 + 2x^2 + 2x \;\; - 3 \\ \hline \qquad\qquad\qquad\qquad\;\; x \;\; - 2 \\ \hline 2x^5 - 3x^4 + 2x^3 + 2x^2 - 3x \\ \qquad\;\; - 4x^4 + 6x^3 - 4x^2 - 4x + 6 \\ \hline 2x^5 - 7x^4 + 8x^3 - 2x^2 - 7x + 6 \end{array}$$

In this expression the succession of signs is $+ - + - - +$, in which there are four changes of sign; that is, one more change of sign than in (1). If an increase in the number of positive roots always brings about at least an equal increase in the number of changes of sign, then an equation in general form cannot have more positive roots than there are changes of sign in its left member. This is the fact, as we now prove.

DESCARTES'S RULE OF SIGNS. *An equation in general form, $f(x) = 0$, has no more real positive roots than $f(x)$ has changes of sign.*

We shall show that if we multiply each member of an equation of degree n by $x - a$, where a is positive, thus forming an equation of degree $n + 1$, the number of changes of sign in the new equation always exceeds the number of changes of sign in the original equation by at least one; that is, the number of changes of sign increases

at least as rapidly as the increase in the number of positive roots when such a multiplication is made.

Let $f(x) = 0$ represent any particular equation of the nth degree. The first sign of $f(x)$ may always be taken as $+$. The remaining signs occur in successive groups of $+$ or $-$ signs which may contain only one sign each. If any term is lacking, its sign is taken to be the same as one of the adjacent signs. Thus the way in which the signs of $f(x)$ may occur is represented in the following table, in which the dots represent an indefinite number of signs. The multiplication of $f(x)$ by $x - a$ is represented schematically, only the signs being given.

	All + signs	All − signs	All + signs	All − signs	Further groups	All − signs
$f(x)$,	$+ \cdots\cdot +$	$- \cdots\cdot -$	$+ \cdots\cdot +$	$- \cdots\cdot -$	$+ \cdots\cdot +$	$- \cdots\cdot -$
$x - a$,						$+ \quad -$
$xf(x)$,	$+ + \cdots +$	$- - \cdots -$	$+ + \cdots +$	$- - \cdots -$	$+ + \cdots +$	$- - \cdots -$
$-af(x)$,	$- \cdots -$	$- + \cdots +$	$+ - \cdots -$	$- + \cdots +$	$+ - \cdots -$	$- + \cdots + +$
$(x - a)f(x)$,	$+ \pm \cdots \pm$	$- \pm \cdots \pm$	$+ \pm \cdots \pm$	$- \pm \cdots \pm$	$+ \pm \cdots \pm$	$- \pm \cdots \pm +$

The $\pm$ sign indicates that either the $+$ or the $-$ sign may occur according to the values of the coefficients and of a. The vertical lines denote where changes of sign occur in $f(x)$. Assuming that all the ambiguous signs are taken so as to afford the least possible number of changes of sign, even then in $(x - a)f(x)$ there is a change of sign at or before each of the vertical lines, and in addition, one to the right of all the vertical lines. Hence as we increase the number of positive roots by one, the number of changes of sign increases at least by one, perhaps by more.

The only possible variation which could occur in the succession of groups of signs in $f(x)$, namely, where the last group consists of $+$ signs, does not alter the validity of the theorem.

Illustration. Let $f(x) = x^5 - 4x^3 - x + 2$, and let $a = 2$.

$$\begin{array}{rl l}
f(x), & 1 + 0 - 4 - 0 - 1 + 2 & \qquad 2 \text{ changes} \\
x - 2, & \qquad\qquad\qquad\quad 1 - 2 & \\
\hline
xf(x), & 1 + 0 - 4 - 0 - 1 + 2 & \\
-2f(x), & \quad - 2 - 0 + 8 + 0 + 2 - 4 & \\
\hline
(x - 2)f(x), & 1 - 2 - 4 + 8 - 1 + 4 - 4 & \qquad 4 \text{ changes}
\end{array}$$

Since $f(-x)=0$ has roots opposite in sign to those of $f(x)=0$ (§ 65), we can state

DESCARTES'S RULE OF SIGNS FOR NEGATIVE ROOTS. *The general equation $f(x)=0$ has no more negative roots than there are changes in sign in $f(-x)$.*

If by Descartes's rule it appears that there cannot be more than a positive roots and b negative roots, and if $a+b<n$, where n is the degree of the equation, then there must be complex roots, at least $n-(a+b)$ in number.

In applying Descartes's rule no signs need be supplied for the missing terms.

EXAMPLES

Obtain all the information possible concerning the roots of each of the following equations by the use of Descartes's rule and by inspection of the constant term.

1. $x^3+3x^2+1=0$.

Solution. $f(x): + + +$, no change; therefore no positive root.

$f(-x): - + +$, one change; therefore not more than one negative root.

Since the equation has three roots, it has one negative and two complex roots.

2. $x^5-x+1=0$.

Solution. $f(x): + - +$, two changes; therefore not more than two positive roots.

$f(-x): - + +$, one change; therefore not more than one negative root.

There are five roots in all and there must be an even number of complex roots. Hence there are three possibilities which may be represented by the following table:

+	−	comp.
2	1	2
1	0	4
0	1	4

Of course only one of these combinations actually occurs, but Descartes's rule does not tell us which one.

Since, however, the constant term, with its sign changed, equals the product of the roots, and since the product of conjugate complex numbers is always positive, the second combination cannot occur; that is, the equation surely has a negative root.

EXERCISES

Obtain all the information possible concerning the roots of each of the following equations by the use of Descartes's rule and by inspection of the constant term:

1. $x^5 + 3x^2 + 2 = 0.$

2. $2x^3 + 1 = 0.$

3. $x^4 + x^2 + 1 = 0.$

4. $x^7 + 6x^2 + 3 = 0.$

5. $3x^3 - x^2 + 2x - 1 = 0.$

6. $x^5 - 1 = 0.$

7. $x^8 + 1 = 0.$

8. $x^5 - x^4 - 2x^3 + 3x^2 + 2x + 1 = 0.$

9. $x^6 + 3x^4 + 2x^2 + 1 = 0.$

10. $5x^5 - 3x - 1 = 0.$

NOTE. In the following exercises n is to be regarded as a positive integer.

11. $x^{2n} + 1 = 0.$

12. $x^{2n} - 1 = 0.$

13. $x^{2n+1} + 1 = 0.$

14. $x^{2n+1} - 1 = 0.$

15. Find the equation whose roots are twice the roots of the equation $x^4 - 2x^3 - 3x + 1 = 0.$

16. Find the equation whose roots are one third the roots of the equation $2x^3 - 6x^2 + 3 = 0.$

17. Find the equation whose roots are equal to the roots of the equation $x^3 + 2x^2 - 8x + 8 = 0$, each multiplied by $-\frac{3}{2}$.

18. Find the equation whose roots are four times the roots of the equation $x^5 - 2x^4 + \frac{1}{64}x - \frac{1}{256} = 0.$

19. Form the equations whose roots are the negatives of the roots of the equations in the four preceding exercises.

20. Transform the equation $x^4 - \frac{3}{2}x^3 + \frac{1}{2}x^2 + 2x - 1 = 0$ by multiplying the roots by the smallest number which will make the coefficients of the transformed equation integers, and the coefficient of the first term unity. Solve the transformed equation, and hence obtain the roots of the original equation.

67. Diminution of the roots of an equation. Before proceeding with the determination of the irrational roots of an equation it is necessary to show how to form an equation whose roots differ from the roots of a given equation by a constant.

Suppose the general equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (A)$$

with the roots $x_1, x_2, \cdots, x_n$, is given, and it is required to find an equation whose roots are less than these numbers by the constant a; that is, we seek an equation which is satisfied by the numbers

$$x_1 - a, \quad x_2 - a, \quad \cdots, \quad x_n - a. \tag{1}$$

Let $x = x' + a$ and consider the equation

$$f(x' + a) = a_0(x' + a)^n + a_1(x' + a)^{n-1} + \cdots + a_n = 0, \tag{2}$$

where x' is the variable. We shall show that this equation is satisfied by the numbers (1). Replacing x' by any of the numbers (1), say, $x_1 - a$, the polynomial in the left member of (2) becomes

$$f(x_1 - a + a) = f(x_1).$$

But $f(x_1) = 0$, since x_1 is a root of the equation $f(x) = 0$. Hence $x_1 - a$ is a root of equation (2). Similarly, all the numbers (1) are roots of equation (2).

To express equation (2) in general form, it is only necessary to remove the parentheses and collect powers of x'. The result may be written as follows:

$$F(x') = A_0x'^n + A_1x'^{n-1} + \cdots + A_n = 0, \tag{3}$$

where the A's are the coefficients which we obtain by collecting like powers of x'. Since the coefficients in this function are different from those in (A), we denote it by a different symbol, $F(x')$.

Illustration. Consider the equation

$$f(x) = x^3 - 6x^2 + 11x - 6 = 0, \tag{4}$$

whose roots are 1, 2, and 3. Let us find the equation whose roots are less by 2 than those of (4). Let $x = x' + 2$, and form the equation $f(x' + 2) = 0$. W obtain

$$f(x' + 2) = (x' + 2)^3 - 6(x' + 2)^2 + 11(x' + 2) - 6 = 0.$$

Simplifying (5), we get $\quad F(x') = x'^3 - x' = 0.$

We see that the roots of this equation, -1, 0, and 1, are less by roots of equation (4), namely, 1, 2, and 3.

We will now derive a method of obtaining the more rapidly than they can be computed by exp in (2). It must be kept in mind that $x' + a$ ar symbols for the same thing; that is,

$$x = x' + a, \quad \text{or}$$

and we may at any time use the notation which is most convenient for us. Since (2) and (3) are identical, we have

$$F(x') = f(x' + a) = f(x).$$

We wish to compute the coefficients in the expression

$$F(x') = A_0 x'^n + A_1 x'^{n-1} + \cdots + A_n. \tag{7}$$

If we divide the right member of (7) by x', we obtain A_n as the remainder. But since $$F(x') = f(x),$$
and $$x' = x - a,$$
the result of dividing $F(x')$ by x' is the same as that of dividing $f(x)$ by $x - a$. Since $f(x)$ is given, we can readily divide it by $x - a$ by the synthetic method, and in this manner find the numerical value of A_n.

The quotient of dividing (7) by x' is $A_0 x'^{n-1} + A_1 x'^{n-2} + \cdots + A_{n-1}$. If we divide this quotient by x', we obtain the coefficient A_{n-1} as a remainder. But this division is precisely equivalent to dividing the quotient of $\dfrac{f(x)}{x-a}$ by $x - a$. Proceeding similarly, we may obtain in order $A_{n-2}, \cdots, A_0$. This method for computing the coefficients of the equation whose roots are less than those of $f(x) = 0$ by the constant a we may express by the following

RULE. *The constant term of the new equation is the remainder after dividing $f(x)$ by $x - a$.*

The coefficient of x' in the new equation is the remainder after dividing the quotient just obtained by $x - a$.

The coefficients of the higher powers of x' are the remainders after dividing the successive quotients obtained by $x - a$.

Illustration. Let us compute by this method the coefficients of the equation whose roots are less by 2 than those of (4). We first divide by $x - 2$ synthetically.

$$\begin{array}{l|l} 1 - 6 + 11 - 6 & 2 \\ \quad + 2 - \;\; 8 + 6 & \\ \hline 1 - 4 + \;\; 3 + 0 & \end{array}$$

Hence 0 is the value of the constant term in the new equation. By § 55 the coefficients of the quotient in this division are the numbers in the last line of the division up to the remainder. Dividing this quotient by $x - 2$ synthetically, we obtain -1 as the next to the last coefficient of the new equation.

$$\begin{array}{l|l} 1 - 4 + 3 & 2 \\ \quad + 2 - 4 & \\ \hline 1 - 2 - 1 & \end{array}$$

The coefficients of the new quotient are 1 and -2, and, performing the next division, we get

$$\begin{array}{l|l} 1-2 & 2 \\ \underline{\;\;+2} & \\ 1+0 & \end{array}$$

This process may be arranged more compactly as follows, where the full-faced type shows the coefficients of the transformed equation:

$$\begin{array}{rrrr|l} 1 & -6 & +11 & -6 & 2 \\ & +2 & -8 & +6 & \\ \hline 1 & -4 & +3 & \mathbf{+0} & \\ & +2 & -4 & & \\ \hline 1 & -2 & \mathbf{-1} & & \\ & +2 & & & \\ \hline \mathbf{1} & \mathbf{+0} & & & \end{array}$$

Hence the new equation is $x'^3 - x' = 0$.

We may now drop the primes and write the new equation,

$$x^3 - x = 0.$$

68. Graphical meaning of the transformation. If an equation has each of its roots decreased by the positive number a, then the graph of the function in the new equation will cross the X axis a units farther to the left than the graph of the old one. In fact, the new graph is just the same as the old one, except that its position is a units to the left. This is expressed by the relation $x' = x - a$, which indicates that the abscissas for points on the new curve are each a units shorter than those for the corresponding points on the old one.

By means of this transformation we may bring any crossing of a graph within one unit of the origin. This corresponds to decreasing the roots of the original equation by a number such that one of the roots of the new equation falls between 0 and 1.

Decreasing the roots by a negative number is equivalent to increasing them and to moving the graph to the right.

EXERCISES

1. Transform the equation $x^3 - 4x^2 + x + 6 = 0$ into an equation whose roots are less by 2 than the roots of the given equation. Plot the function forming the left member of each equation.

2. Transform the equation $x^4 + x^3 - 5x^2 + 3x = 0$ into an equation whose roots are greater by 1 than the roots of the given equation. Plot the function forming the left member of each equation.

Transform each of the following equations into one whose roots are less by the number opposite than the roots of the given equation:

3. $x^3 - 15x^2 + 7x + 125 = 0$, 5.

4. $x^4 - 2x^2 + 1 = 0$, .2.

5. $x^4 + 6x^3 + 10x^2 + x - 1 = 0$, -1.

6. $2x^3 - 5x^2 + x + 2 = 0$, $\frac{1}{2}$.

7. $16x^4 - 13x^2 + 9 = 0$, 1.5.

8. $x^4 - 1.5x^2 + 2x - 2.5 = 0$, 2.

9. Transform the equation $36x^3 - 108x^2 + 107x - 35 = 0$ into an equation whose roots are less by 1 than the roots of the given equation. Solve the transformed equation and thus determine the roots of the given equation.

10. Transform the equation $16x^4 - 72x^2 - 64x - 15 = 0$ into an equation whose roots are greater by .5 than the roots of the given equation. Solve the transformed equation and thus determine the roots of the given equation.

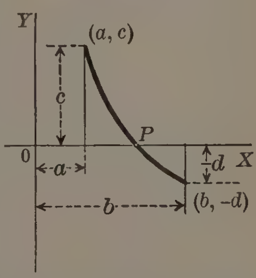
11. How much must the roots of the equation $x^4 - 8x^3 + 9x^2 + 38x - 40 = 0$ be diminished in order that the sum of the roots of the transformed equation shall be 0? Find the transformed equation.

Hint. The sum of the roots must be diminished by 8.

12. How much must the roots of the equation $x^3 + 4x^2 - 3x + 7 = 0$ be diminished in order that the coefficient of x in the transformed equation shall be 0? Find the transformed equation.

Hint. Decrease the roots by h, and determine h so that the coefficient of x' is zero.

69. Location principle. If in plotting a function $y = f(x)$ the value $x = a$ gives the corresponding value of y positive and equal to c, while the value $x = b$ gives the corresponding value of y negative, say, equal to $-d$, then the point (a, c) on the curve is above the X axis, and the point $(b, -d)$ on the curve is below the X axis. If the curve is unbroken, it must then cross the X axis at least once between the values $x = a$ and $x = b$, and hence the equation $f(x) = 0$ must have a root between these values of x. The shorter we can determine this interval between a and b, the more accurately we can find the root of the equation. Horner's Method of



approximation, which we shall explain in the next section, is nothing but an ingenious process for making the interval in which we know a root must exist as small as we wish. We have throughout this text assumed the property of unbrokenness or continuity of the graph of $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$.

This geometric assumption may be expressed in algebraic language in the following

Location principle. *When for two real unequal values of x, say, $x = a$ and $x = b$, the values of $y = f(x)$ have opposite signs, the equation $f(x) = 0$ has a real root between a and b.*

The interval between $x = a$ and $x = b$ we shall call the **location interval.**

70. Horner's Method of approximating irrational roots. We are now in a position to determine the real roots of an equation to any desired degree of accuracy.

It is assumed that all rational roots have been found by the methods of § 64, and that all the roots which remain are either irrational or complex.

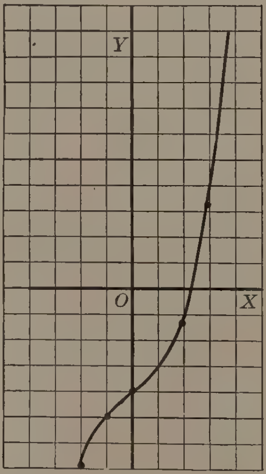
Consider the equation

$$x^3 + 3x - 20 = 0. \qquad (1)$$

Let us find its real roots to two decimal places.

First form a table of values for the function $y = x^3 + 3x - 20$, and plot the function.

x	0	1	2	3	-1
y	-20	-16	-6	$+16$	-24



By Descartes's rule it appears that (1) has no negative root. By the location principle it appears that there is a root between 2 and 3. The whole point of Horner's Method consists in decreasing the roots of the successive equations which we meet in the course of the process by the lesser of the two numbers which bound the location interval.

Here we decrease the roots of equation (1) by 2, as follows:

$$\begin{array}{rrrr|l} 1 & +0 & +3 & -20 & \underline{2} \\ & +2 & +4 & +14 & \\ \hline 1 & +2 & +7 & -6 & \\ & +2 & +8 & & \\ \hline 1 & +4 & +15 & & \\ & +2 & & & \\ \hline 1 & +6 & & & \end{array}$$

The resulting equation is

$$x^3 + 6x^2 + 15x - 6 = 0. \qquad (2)$$

We know that equation (2) has a root between 0 and 1, since equation (1) has a root between 2 and 3. From the graph we can estimate in tenths the position of this root. Having made an estimate, say .3, it is necessary to verify it and to determine by synthetic division precisely between which tenths the root lies. Thus, trying .3, we obtain

$$\begin{array}{rrrr|l} 1 & +6.0 & +15.00 & -6.00 & \underline{.3} \\ & +0.3 & +1.89 & +5.07 & \\ \hline 1 & +6.3 & +16.89 & -0.93 & \end{array}$$

In the computation by Horner's Method it is usually unnecessary to preserve more decimal places than are called for in the root which is sought. That is, we avoid carrying through the process figures which have no effect on the result. In the present example we shall write down the decimals correct to two places. For instance, in the last multiplication by .3 above, the product is 5.067, but we write only 5.07, the approximate value to two decimal places. When the figure in the third decimal place is 5 or more, we add 1 to the figure in the second decimal place; when it is less than 5, we drop it. If three decimal places had been required in the root, we would have preserved the decimals in the computation correct to three places. This method of shortening the computation is sufficiently accurate except in rare cases, where the remainder by synthetic division is so near 0 that its sign would be changed if all the figures of the decimals were retained. In such a case we would perform the synthetic division retaining all figures of the decimals.

Since the remainder in this synthetic division is negative, it appears that for $x = .3$ the curve is below the X axis, and that the root is greater than .3. But we are not justified in assuming that the root is between .3 and .4 until we have substituted .4 for x. This we proceed to do.

$$\begin{array}{rrrr|l} 1 & +6.0 & +15.00 & -6.00 & \underline{.4} \\ & +0.4 & +2.56 & +7.02 & \\ \hline 1 & +6.4 & +17.56 & +1.02 & \end{array}$$

Since the remainder is positive for $x = .4$, the location principle shows that (2) has a root between .3 and .4; that is, (1) has a root between 2.3 and 2.4.

To find the root to two decimal places, decrease the roots of (2) by .3, the lesser of the two numbers between which the root of (2) is now known to lie. The new equation has a root between 0 and .1.

This process is performed as follows:

$$\begin{array}{llll|l}
1 & +6.0 & +15.00 & -6.00 & \underline{.3} \\
 & +0.3 & +\ \ 1.89 & +5.07 & \\
\hline
1 & +6.3 & +16.89 & \mathbf{-0.93} & \\
 & +0.3 & +\ \ 1.98 & & \\
\hline
1 & +6.6 & \mathbf{+18.87} & & \\
 & +0.3 & & & \\
\hline
\mathbf{1} & \mathbf{+6.9} & & &
\end{array}$$

Thus the new equation is

$$x^3 + 6.9x^2 + 18.87x - .93 = 0. \tag{3}$$

This equation has a root between 0 and .1. We can find an approximate value of the hundredths' place of the root by solving the linear equation $18.87x - .93 = 0$, obtained from (3) by dropping all but the term in x and the constant term.

Thus $$x = \frac{.93}{18.87} = .04.$$

This suggestion must be verified by synthetic division to determine between what hundredths a root of (3) actually lies.

$$\begin{array}{llll|l}
1 & +6.90 & +18.87 & -0.93 & \underline{.04} \\
 & +0.04 & +\ \ 0.28 & +0.77 & \\
\hline
1 & +6.94 & +19.15 & -0.16 &
\end{array}$$

Thus the curve is below the X axis at $x = .04$ and hence the root is greater than .04. We must not assume that the root is between .04 and .05 without determining that the curve is above the X axis at $x = .05$.

$$\begin{array}{llll|l}
1 & +6.90 & +18.87 & -0.93 & \underline{.05} \\
 & +0.05 & +\ \ 0.35 & +0.96 & \\
\hline
1 & +6.95 & +19.22 & +0.03 &
\end{array}$$

Thus the curve is above the X axis at $x = .05$. By the location principle, (3) has a root between .04 and .05; that is, (1) has a root between 2.34 and 2.35. Hence the root to two decimal places is 2.34.

The preceding computation may be arranged more compactly as follows:

$$
\begin{array}{l}
1 + 0 + 3 - 20 \,\lfloor 2 \\
\;\;\; + 2 + 4 + 14 \\
\hline
1 + 2 + 7 \,|\, - 6 \\
\;\;\; + 2 + 8 \\
\hline
1 + 4 \,|\, + 15 \\
\;\;\; + 2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
1 + 6.0 + 15.00 - 6.00 \,\lfloor .4 \\
\;\;\;\;\;\; .4 + 2.56 + 7.02 \\
\hline
1 + 6.4 + 17.56 + 1.02
\end{array}
$$

$$x = \frac{.93}{18.87} = .04.$$

$$
\begin{array}{l}
\mathbf{1 + 6.0 + 15.00 - 6.00} \,\lfloor .3 \\
\;\;\; + .3 + 1.89 + 5.07 \\
\hline
1 + 6.3 + 16.89 \,|\, - .93 \\
\;\;\; + .3 + 1.98 \\
\hline
1 + 6.6 \,|\, + 18.87 \\
\;\;\;\;\;\; .3 \\
\hline
\mathbf{1 + 6.9 + 18.87 - .93}
\end{array}
\qquad
\begin{array}{l}
1 + 6.90 + 18.87 - .93 \,\lfloor .04 \\
\;\;\; + .04 + .28 + .77 \\
\hline
1 + 6.94 + 19.15 - .16 \\
\\
1 + 6.90 + 18.87 - .93 \,\lfloor .05 \\
\;\;\; + .05 + .35 + .96 \\
\hline
1 + 6.95 + 19.22 + .03
\end{array}
$$

Root $= 2.34 +$.

The foregoing process affords the following

RULE. *Obtain all possible information about the roots by Descartes's rule.*

Plot the function. Apply the location principle to determine between what consecutive positive integral values a root lies.

Decrease the roots of the equation by the lesser of the two numbers which bound a location interval.

Estimate from the plot the nearest tenth to which the desired root of the new equation lies, and determine by synthetic division precisely the successive tenths between which the root lies.

Decrease the roots of the new equation by the lesser of the two numbers which bound its tenths' location interval, and estimate the root of the resulting equation to the nearest hundredth by solving the linear equation formed by dropping all but the last two terms of the equation.

Determine precisely by synthetic division the hundredths' location interval.

Proceed similarly to find the root to as many places as may be desired.

The sum of the integral, tenths, and hundredths values obtained in the foregoing process is the approximate value of the root.

To find the negative roots of an equation $f(x) = 0$, determine the positive roots of $f(-x) = 0$ and change their signs.

When all the roots of an equation in the p-form are real, a check to the accuracy of the computation may be found by adding the roots together. The result should be the coefficient of the second term with its sign changed.

Sometimes an equation has roots so nearly equal that the table of values formed for integral values of x gives no information as to whether there are roots between two consecutive integers or not. For instance, the table of values for the equation $x^3 + 17x^2 - 46x + 29 = 0$ does not tell us whether the equation has three real roots or only one. In such a case we might form a table, using values of x differing by .1 or .01, and thus locate the root between two successive tenths or hundredths. But since such equations occur very rarely in practice, and since the calculus affords a very simple method of determining a number between the roots if they are real and distinct, the complete discussion of this case will not be given here.*

EXERCISES

1. Find to two decimal places a positive root of $x^3 + 3x^2 - 2x - 1 = 0$.

2. Find to two decimal places a positive root of $x^3 - 6x^2 + 10x - 9 = 0$.

3. Find to two decimal places all the real roots of $2x^4 - 4x^3 + 3x^2 - 1 = 0$.

4. Find to two decimal places all the real roots of $x^3 + 4x^2 - 7 = 0$.

5. Find to two decimal places all the real roots of $x^4 - 4x^3 + 14x^2 - 4x - 447 = 0$.

6. Find to three decimal places a positive root of $x^3 - 9x^2 + 25x - 18 = 0$.

7. Find to three decimal places a positive root of $x^3 - 2x^2 + 2x - 101 = 0$.

8. Find to three decimal places all the real roots of $3x^4 - 5x^3 = 31$.

9. Find exactly a real root of $4x^3 + 23x^2 - x - 377 = 0$, and show that the other roots are imaginary.

* See Hawkes, Advanced Algebra, p. 200.

10. Show that the equation $x^3 - 7x + 7 = 0$ has two roots between 1 and 2 and one negative root. Find each of the roots to three decimal places.

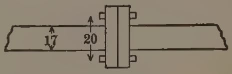
11. Solve exercise 11, p. 96, getting the result to two decimal places.

12. Solve exercise 13, p. 96, getting the result to two decimal places.

13. If a wooden simple beam x inches square and of 12 feet span carries a load of 300 pounds at the middle when it is also subject to a longitudinal tension of 2000 pounds, the allowable tensile strength being 1000 pounds per square inch, the safe value of x is given by the equation $x^3 - 2x - 64.8 = 0$. Find to two decimal places the size of the beam.

14. A beam of span l, fixed at one end and resting on a support at the other end, is subjected to a uniformly distributed load. The point of maximum deflection for a safe load is given by the equation $8x^3 - 9lx^2 + l^3 = 0$, where x is the distance from the supported end. Show that $x = .4215\,l$.

15. A hollow cylindrical shaft 17 inches in outside, and 11 inches in inside, diameter is to be coupled by 12 bolts placed with their centers 20 inches from the axis. The proper diameter of the bolts is given by the equation $d^4 + 3200\,d^2 - 337.6\,d - 13{,}500 = 0$. Find the diameter of the bolts to one decimal place.



16. A wooden column x inches square and 12 feet long, having fixed ends, is to carry an axial load of 50 net tons with a factor of safety of 10. The size of the column is given by the equation $x^4 - 125x^2 = 10{,}368$. Find the size of the column to one decimal place.

17. If the column in the preceding exercise has round ends, its size is given by the equation $x^4 - 125x^2 = 41{,}472$. Find the size of the column to one decimal place.

18. In exercise 16, if the eccentricity of the load is 2.5 inches, the size of the column is given by the equation $x^4 - 125x^2 - 1875x - 10{,}368 = 0$. Find the size of the column to one decimal place.

19. The Gas Equation of Van der Waals is $\left(p + \frac{a}{v^2}\right)(v - b) = 1$, where v is the volume of the gas, p the pressure, and a and b are

constants depending on the gas. For carbonic acid gas $a = .00874$ and $b = .0023$. Find the value of v to two decimal places when $p = 1$.

HINT. Reduce the equation to one of third degree in v with numerical coefficients and multiply the roots by 10 before solving.

20. The following equation occurs in the theory of chemical actions *: $\frac{1}{4}x^3 + \frac{2}{5}x^4 + \frac{4}{6}x^5 = 1.106$. Find the value of x to two decimal places.

21. The cubical coefficient of thermal expansion of paraffin is .000584 per degree centigrade. If t be the temperature on the centigrade scale, the linear coefficient of expansion of the paraffin is to be found from

$$3a + 6a^2t + 3a^3t^2 = .000584.$$

Find to two decimal places the linear coefficient of expansion a at 30° C.

22. An empirical formula for the volume of one gram of water at temperature t degrees centigrade is

$$v = 1 - .00009417\,t + .000001449\,t^2 + .0000005985\,t^3,$$

where v is the volume in cubic centimeters. Find correct to tenths of a degree the temperature at which the volume of 1 gram of water will be 1.0002 cubic centimeters.

71. Solution of the cubic. In § 19 exact expressions for the roots of the quadratic equation $ax^2 + bx + c = 0$ were found in terms of its coefficients, a, b, and c. The only process of approximation needed in order to find an irrational root of such an equation is that of extracting the square root. Hence it is never necessary to use Horner's Method to find the roots of a quadratic equation.

In this section we shall see that it is possible to find an exact expression for the roots of the cubic equation in terms of its coefficients, and that the formulas obtained may be used in certain cases as a substitute for Horner's Method. A similar but more laborious solution for the equation of the fourth degree exists, but will not be given here. It is, however, impossible to obtain any general solution of equations of higher degree than the fourth by means of algebraic operations.

The cubic equation in the p-form is

$$x^3 + p_1x^2 + p_2x + p_3 = 0. \tag{1}$$

* J. L. R. Morgan, Elements of Physical Chemistry, 4th ed., p. 506.

If we increase each of the roots of this equation by $\frac{p_1}{3}$, in order to remove the term in x^2, we obtain the equation

$$x^3 + \left(p_2 - \frac{p_1^2}{3}\right)x + \frac{2\,p_1^3}{27} - \frac{p_1 p_2}{3} + p_3 = 0, \tag{2}$$

lacking the term in x^2, and which we may write in the form

$$x^3 + px + q = 0, \tag{3}$$

where $$p = p_2 - \frac{p_1^2}{3}, \text{ and } q = \frac{2\,p_1^3}{27} - \frac{p_1 p_2}{3} + p_3. \tag{4}$$

The cubic in form (3) we can solve as follows: Let

$$x = y + z. \tag{5}$$

This amounts to replacing the single variable x by the two variables y and z. We deliberately complicate the problem in this way, because we shall obtain a relation between y and z which will enable us to find the values of the two more easily than we could determine the value of x alone.

Substituting (5) in (3),

$$(y+z)^3 + p(y+z) + q = y^3 + z^3 + (3\,yz + p)(y+z) + q = 0. \tag{6}$$

Having introduced an extra variable, we are at liberty to impose a condition on y and z.

Let $$3\,yz + p = 0, \quad \text{or} \quad yz = \frac{-p}{3}. \tag{7}$$

Then (6) will reduce to $\quad y^3 + z^3 = -q,$

and from (7) we have $\quad y^3 z^3 = -\frac{p^3}{27}.$

By reason of the relations between the roots and the coefficients of a quadratic equation (§ 24), it appears that y^3 and z^3 must be roots of the quadratic equation whose coefficients are 1, q, and $-\frac{p^3}{27}$; that is, of the equation

$$t^2 + qt - \frac{p^3}{27} = 0. \tag{8}$$

Solving (8), we find

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = A; \quad z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = B.$$

Hence $y = \sqrt[3]{A},\ \omega\sqrt[3]{A},\ \omega^2\sqrt[3]{A}$; $z = \sqrt[3]{B},\ \omega\sqrt[3]{B},\ \omega^2\sqrt[3]{B}$, where ω represents one of the complex cube roots of 1 (exercise 27, p. 86).

The roots of (3) would at first sight seem to be nine in number, namely, the values which we obtain by adding each of the three values of y to each of those of z. But reference to (7) reminds us

that the product yz must be real; hence all of these nine values are ruled out except the following, which are the roots of (3):

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B};\quad x_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B};\quad x_3 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}.$$

These expressions are called Cardan's Formulas for the solution of the cubic.

When the value of $\frac{q^2}{4} + \frac{p^3}{27}$ is positive we can extract its square root and compute the values of A and B readily. In this case we find but one real root, the other two being complex. If the value of $\frac{q^2}{4} + \frac{p^3}{27}$ is negative, in which case there are three real roots, it is necessary, in order to solve the equation by this method, to extract the cube root of a complex number. This is more laborious than it is to find the roots by Horner's Method. Consequently, we shall use these formulas only when $\frac{q^2}{4} + \frac{p^3}{27} > 0$. In this case there is only one real root.

If an equation of the third degree in p-form has three real roots and they have been found by Horner's Method, we have seen that the work may be checked by adding them together. The result should be the coefficient of the term in x^2 with its sign changed. If the equation has only one real root, this check is not available, and the result of Horner's Method may be checked by the method of this section.

The foregoing method of solving the cubic is due to the Italian, Tartaglia, but was first published by Cardan in 1545. At this time the operations with complex numbers were imperfectly understood, and every effort was made by mathematicians to avoid them. It must have been not a little irritating for the early algebraists to realize that the only case in which Cardan's Formulas solve a cubic without extracting a root of a complex number is that in which the equation has a pair of complex roots. To find the three real roots of a cubic when they are irrational, the case in which they were chiefly interested, the cube roots of complex numbers are necessary.

EXERCISES

Solve the following cubics, using Cardan's Formulas:

1. $x^3 + 3x^2 + 3x + 2 = 0.$

2. $x^3 - 11x + 20 = 0.$

3. $4x^3 + 2x^2 - 1 = 0.$

4. $2x^3 - 9x^2 + 2x + 30 = 0.$

Find to two decimal places the real root of each of the following cubics, using the tables (pp. 215–217) to evaluate the radicals:

5. $x^3 + 3x - 20 = 0.$

6. $x^3 - x - 33 = 0.$

7. $x^3 - 8x - 24 = 0.$

8. $x^3 + 3x^2 + 5x - 17 = 0.$

9. $2x^3 + 12x^2 + 27x - 68 = 0.$

10. $x^3 - 9x^2 + 25x - 18 = 0.$

72. Graphical solution of the quadratic equation. When the real roots of an equation are desired to no more than one decimal place, the equation may be solved graphically. Take, for example, the equation

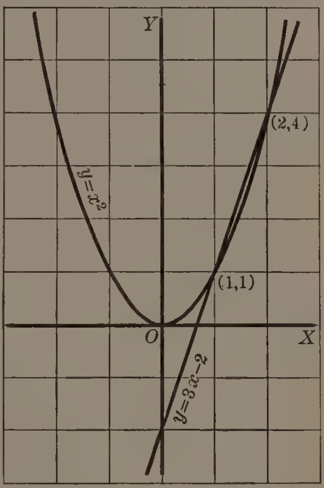
$$x^2 - 3x + 2 = 0. \qquad (1)$$

We seek the values of x which satisfy this equation.

Consider $\qquad y = x^2, \quad \text{and} \quad y = 3x - 2. \qquad (2)$

If we solve these equations, we find two values of x such that the corresponding values of y in equations (2) are equal to each other, since for these values of x we would have

$$x^2 = 3x - 2, \text{ or } x^2 - 3x + 2 = 0.$$



To solve this problem graphically we plot equations (2) on the same axes and note the values of x where the straight line $y = 3x - 2$ intersects the parabola $y = x^2$. These are the values of x which afford equal values for y in equations (2), the coördinates of both curves being identical at a point of intersection.

In the adjacent graph the abscissas of the points of intersection are 1 and 2 respectively. Hence these numbers are the roots of (1).

The advantage of this graphical method lies in the fact that for all quadratic equations in p-form the first equation of (2) is the same, and hence the parabola may be drawn once for all in ink. This leaves only the necessity of drawing in pencil one straight line for each solution and noting the points of intersection.

EXERCISES

Solve graphically the following quadratics:

1. $x^2 + 2x - 8 = 0.$
2. $x^2 + x - 3\frac{3}{4} = 0.$
3. $x^2 - 7x + 12.25 = 0.$
4. $x^2 - 3x - 2 = 0.$
5. $x^2 + 4x + 3 = 0.$
6. $2x^2 - 6x + 1 = 0.$
7. $3x^2 + 5x + 2 = 0.$
8. $x^2 + .3x - 2.4 = 0.$
9. $x^2 - 3.7x + 3.3 = 0.$
10. $x^2 - 3x - 3.5 = 0.$

73. Graphical solution of the cubic equation. It is assumed in what follows that the term in x^2 has been removed from the cubic in p-form by increasing each of the roots by $\frac{p_1}{3}$, leaving the equation in the form

$$x^3 + px + q = 0. \tag{1}$$

This transformation should be performed by synthetic division, as in § 67.

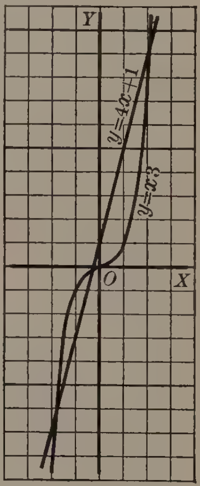
We then plot the curve $y = x^3$ and the straight line $y = -(px + q)$ on the same axes, and note the x distances of their points of intersection. These will be the real roots of (1). The plot of the curve $y = x^3$ should be made carefully on a large scale in ink, so that the line may be drawn in pencil, as in the preceding section. In this way the same curve may serve for many problems. This method gives only the real roots of the cubic, and there will be one or three real roots according as the line $y = -(px + q)$ cuts the curve $y = x^3$ in one or three points.

EXAMPLE

Solve graphically $\quad x^3 + 6x^2 + 8x - 1 = 0. \qquad (1)$

Solution. Here $p_1 = 6$. We must then increase the roots by 2.

$$\begin{array}{l|l}
1 + 6 + 8 - 1 & \underline{-2} \\
\underline{\quad - 2 - 8 + 0} & \\
1 + 4 + 0 - 1 & \\
\underline{\quad - 2 - 4} & \\
1 + 2 - 4 & \\
\underline{\quad - 2} & \\
1 + 0 &
\end{array}$$



The transformed equation is

$$x^3 - 4x - 1 = 0. \tag{2}$$

Plot the equations

$$y = x^3$$

and

$$y = 4x + 1$$

on the same axes.

The abscissas of the points of intersection are approximately 2.1, $-.3$, and -1.8. Hence these are approximately the roots of (2). Since the roots of (1) were increased by 2 to form (2), we obtain the numbers .1, -2.3, and -3.8 as the approximate values of the roots of (1).

EXERCISES

Find graphically the real roots of the following cubics:

1. Equation of exercise 10, p. 132.
2. Equation of exercise 4, p. 128.
3. Equation of exercise 5, p. 132.
4. Equation of exercise 8, p. 132.
5. $x^3 + x - 20 = 0.$
6. $x^3 + 3x^2 - 2x - 1 = 0.$
7. $x^3 - 6x^2 + 5x + 11 = 0.$
8. $x^3 - 9x^2 - 2x + 101 = 0.$

74. Derived function of the cubic. Let us consider the cubic function

$$f(x) = x^3 + ax^2 + bx + c.$$

In this expression replace x by $x + h$, where h is a real number. We then have

$$f(x+h) = (x+h)^3 + a(x+h)^2 + b(x+h) + c.$$

Expanding the terms of this function by the Binomial Theorem and collecting like powers of h, we have

$$f(x+h) = x^3 + ax^2 + bx + c + (3x^2 + 2ax + b)h + (3x + a)h^2 + h^3.$$

The coefficient of h in this expansion is called the **first derivative** of $f(x)$, and is symbolized by $f'(x)$. If we write $f(x)$ and $f'(x)$ in separate lines, we can observe the relation which they bear to each other.

$$f(x) = x^3 + ax^2 + bx + c.$$
$$f'(x) = 3x^2 + 2ax + b.$$

We see that the first term of $f'(x)$ has as its coefficient 3, which is the exponent of the first term of $f(x)$, while its exponent, 2, is one less than the exponent of the first term of $f(x)$. We may obtain the second and third terms of $f'(x)$ from those of $f(x)$ in a similar manner. The last term, c, of $f(x)$ may be regarded as cx^0; then the corresponding term of $f'(x)$ is $0 \cdot c \cdot x^{-1} = 0$.

This procedure suggests the following rule for finding the derivative of $f(x)$, the general validity of which we shall establish in the next section.

RULE. *If the kth term of $f(x)$ is multiplied by its exponent, and its exponent is decreased by one, the result is the kth term of $f'(x)$.*

EXAMPLE

Find the first derivative of $f(x) = x^3 + 3x^2 - 7x + 4.$

Solution. By the rule we obtain

$$f'(x) = 3x^2 + 6x - 7.$$

EXERCISES

Find the first derivatives of the following functions:

1. $8x^3 - 2x^2 + x - 1$.
2. $x^3 + x^2 + x + 1$.
3. $7x^3 - 6x - 5$.
4. $-3x^3 + 2x^2 - 8$.
5. $a_0x^3 + a_1x^2 + a_2x + a_3$.
6. $\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}$.
7. $4x(x^2 - 2x + 3)$.
8. $(x-1)^3$.
9. $(x^2 + 3)(x - 2)$.
10. $2(x+1)^2(x-1)$.
11. If $f(x) = x^3 + 3x^2 + 6x + 6$, show that $f'(x) = f(x) - x^3$.
12. If $f(x) = x^3 + 6x^2 + 12x + 8$, show that $3f(x) = (x+2)f'(x)$.

75. Derivative of a polynomial. Instead of confining ourselves to the cubic or biquadratic function, let us now consider the polynomial of order n,
$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n.$$

Replace x by $x + h$ in this function, expand each term by the Binomial Theorem, collect the terms free from h, and also those containing h to the first power. We then obtain

$$\begin{aligned}
f(x+h)& \\
&= a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n \\
&= a_0[x^n + nx^{n-1}h + \tfrac{1}{2}n(n-1)x^{n-2}h^2 + \cdots + h^n] \\
&\qquad + a_1[x^{n-1} + (n-1)x^{n-2}h \\
&\qquad + \tfrac{1}{2}(n-1)(n-2)x^{n-3}h^2 + \cdots + h^{n-1}] \\
&\qquad + \cdots + a_{n-1}(x+h) + a_n \\
&= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \\
&\qquad + [a_0nx^{n-1} + a_1(n-1)x^{n-2} + \cdots + a_{n-1}] \cdot h \\
&\qquad + F(x) \cdot h^2 + F_1(x) \cdot h^3 + \cdots + h^n \\
&= f(x) + f'(x)h + F(x) \cdot h^2 + F_1(x)h^3 + \cdots + h^n. \qquad (1)
\end{aligned}$$

In this expansion the coefficient of h is called the first derivative of $f(x)$. It is symbolized by $f'(x)$.

Writing $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$

and $$f'(x) = a_0nx^{n-1} + a_1(n-1)x^{n-2} + \cdots + a_{n-1},$$

we see that $f'(x)$ may be obtained from $f(x)$ by the rule stated in the preceding section.

The successive terms of the expansion (1) which contain h to powers higher than the first are not written out in detail. These coefficients are here represented by $F(x)$, etc.

EXERCISE

Denoting by $f''(x)$ the first derivative of $f'(x)$, show that $F(x)$, the coefficient of h^2 in the expansion (1), is equal to $\frac{f''(x)}{2}$. The expression $f''(x)$ is called the second derivative of $f(x)$.

76. Double roots. The expansion (1) in the preceding section may be written in the following form by replacing x by x_1, h by $x - x_1$, and recalling the result of the preceding exercise:

$$f(x) = f(x_1) + f'(x_1)(x - x_1) \\ + \tfrac{1}{2}f''(x_1)(x - x_1)^2 + \cdots + (x - x_1)^n. \quad (2)$$

An inspection of (2) shows us that if x_1 is such a number that $f(x_1) = 0$, and at the same time $f'(x_1) = 0$, then the first two terms of (2) vanish, leaving all the successive terms divisible by $(x - x_1)^2$. That is, if x_1 is a root of $f(x) = 0$ and of $f'(x) = 0$, but not of $f''(x) = 0$, it is a double root of $f(x) = 0$. This is equivalent to the statement that if $x - x_1$ is a common factor of $f(x)$ and $f'(x)$ but is not a factor of $f''(x)$, then x_1 is a double root of $f(x) = 0$. Hence we have the

RULE. *The number x_1 is a double root of the equation $f(x) = 0$ if $f(x_1) = 0$, $f'(x_1) = 0$, and $f''(x_1) \neq 0$.*

In the above exercise it was determined that the coefficient $F(x)$ of h^2 in (1) is $\frac{f''(x)}{2}$. It may also be shown that the coefficient of h^k in (1) is $\frac{f^{(k)}(x)}{k!}$, where $f^{(k)}(x)$ is the kth derivative of $f(x)$. We may, then, write (2) in the form

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{(x - x_1)^2}{2}f''(x_1) \\ + \cdots + \frac{(x - x_1)^k}{k!}f^{(k)}(x_1) + \cdots + (x - x_1)^n.$$

From this expansion we obtain the rule for finding an r-fold root of $f(x) = 0$.

RULE. *If a number is a root of $f(x) = 0$ and of its first $r - 1$ derivatives, but is not a root of the rth derivative, each set equal to zero, it is an r-fold root of $f(x) = 0$.*

In computing the successive derivatives of $f(x)$, it must be kept in mind that the derivative of any constant term is zero.

EXAMPLES

1. Find a double root of the equation

$$16x^3 - 12x^2 + 1 = 0.$$

Find also the other root.

Solution.

$$f(x) = 16x^3 - 12x^2 + 1,$$
$$f'(x) = 48x^2 - 24x = 24x(2x - 1),$$
$$f''(x) = 96x - 24.$$

We see that $\frac{1}{2}$ is a root of $f'(x) = 0$. It is also a root of $f(x) = 0$, since $f(\frac{1}{2}) = 2 - 3 + 1 = 0$. But $\frac{1}{2}$ is not a root of $f''(x) = 0$. Hence $\frac{1}{2}$ is a double root of $f(x) = 0$.

Since the product of the roots of $f(x) = 0$ is $-\frac{1}{16}$, the other root, r, must be such that $\frac{1}{2} \cdot \frac{1}{2} \cdot r = -\frac{1}{16}$. Hence $r = -\frac{1}{4}$.

Therefore the three roots of $f(x) = 0$ are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}$.

2. Show that 1 is a triple root of the equation

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = 0,$$

and find the other root.

Solution.

$$f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2,$$
$$f'(x) = 4x^3 - 15x^2 + 18x - 7,$$
$$f''(x) = 12x^2 - 30x + 18 = 6(x - 1)(2x - 3),$$
$$f'''(x) = 24x - 30.$$

We see that 1 is a root of $f''(x) = 0$. Also 1 is a root of $f(x) = 0$ and $f'(x) = 0$, since $f(1) = 0$ and $f'(1) = 0$. But 1 is not a root of $f'''(x) = 0$, since $f'''(1) \neq 0$. Hence 1 is a triple root of $f(x) = 0$.

Since the product of the roots of $f(x) = 0$ is 2, the other root is 2.

EXERCISES

Find the double roots and other roots of the following equations:

1. $x^3 - x^2 - 5x - 3 = 0.$

2. $27x^3 - 9x + 2 = 0.$

3. $2x^3 - 15x^2 + 24x + 16 = 0.$

4. $x^3 + 8x^2 + 20x + 16 = 0.$

5. $16x^3 - 60x^2 + 125 = 0.$

6. $63x^3 + 321x^2 + 469x + 147 = 0.$

Find the triple roots and other roots of the following equations:

7. $x^4 - 2x^3 + 2x - 1 = 0.$

8. $2x^4 + 11x^3 + 18x^2 + 4x - 8 = 0.$

9. $16x^4 - 24x^2 + 16x - 3 = 0.$

10. $3x^4 - 32x^3 + 96x^2 - 256 = 0.$

11. Show that 1 is a fourfold root of the equation $2x^5 - 5x^4 + 10x^2 - 10x + 3 = 0$, and find the other root.

12. Show that -1 is a fivefold root of the equation $x^6 + 3x^5 - 10x^3 - 15x^2 - 9x - 2 = 0$, and find the other root.

77. Error in computation. Suppose the values of $f(x)$ are to be computed by substituting values of x which are the result of measurement and hence not known exactly. By means of the derivative we can find the approximate error in the function when the error in x is known, provided that error is small.

Consider, for example, the expression for the volume of a cube in terms of one of its edges, $V = x^3$. If we could measure the edge with perfect correctness, we could find accurately the volume of the cube; but when our rule seems to read, say, 2.25 inches, we know that there may be a slight error in the reading, due to slight inaccuracies in the rule, our vision, and our method of using the rule.

Let the measured value be x_0 and let the small error be denoted by h. Of course we do not ever know just how great h is. We may usually assume, however, that it does not exceed some definite small number. Then letting $x = x_0 + h$, and expanding the function $V = f(x) = x^3$ by formula (1) of the preceding section, we obtain

$$V = f(x_0 + h) = (x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3.$$

Now since h is small, h^2 and h^3 will be much smaller, and may be neglected, as they would not affect the result appreciably. From this expression it appears that the value of the volume differs from the value of x_0^3 by $3x_0^2h$, if we neglect the last two terms. Hence, if we assume that $h = .02$, and $x_0 = 2.25$, the approximate error for V is $3 \cdot (2.25)^2 \cdot .02 = .3$ cubic inches.

In general, let x_0 be the measured value, and h the error of the measurement. We may write (§ 75)

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \cdots.$$

Here $f(x_0)$ is the value of the function if our measurement were correct, while the approximate error in our result, omitting all terms containing powers of h higher than the first, is $h \cdot f'(x_0)$. We may now state the following

RULE. *To find the approximate error in the function $f(x)$ due to a small error, h, in the measurement x_0, multiply h by $f'(x_0)$.*

EXERCISES

1. The edge of a cube is found by measurement to be 3.2 inches. Find the approximate error in the computed volume due to an error of $\frac{1}{20}$ of an inch in measuring the edge.

2. If the diameter of a sphere is found by measurement to be 10.3 inches, find the approximate error in the computed volume due to an error of .1 of an inch in measuring the diameter. (Take $\pi = \frac{22}{7}$.)

3. The height of a cylindrical column is known to be 10 feet. What is the approximate error in the volume computed from a diameter measurement of 50 inches if this measurement is a half inch in error?

4. A surveyor measures a square field with a 50-foot chain which is 1 inch too long and finds the area to be $62\frac{1}{2}$ acres. Find the area of the field in acres correct to 2 decimal places and show that the amount neglected does not affect the second decimal place.

5. Find the approximate error in the function $x^3 - 2x^2 + x - 1$ due to an error of .03 in a value 1.25 taken for x.

6. Find the approximate error in the function $rx^3 - 2r^2x^2$ if r is known to be .1 and the value of x, 8.1, is inaccurate by .1.

7. The diameter of a right circular cylinder whose altitude is 5 feet is measured and found to be 8.2 inches, but the measurement is inaccurate by .1 of an inch. Find the approximate error in computing the total surface.

8. A right circular cylinder is capped by a hemisphere. The height of the cylinder is 50 inches. Its diameter is found by measurement to be $10\frac{3}{8}$ inches. Find the approximate error in computing the total surface of the solid from a diameter measurement which is $\frac{1}{16}$ of an inch in error.

9. A Norman window is in the shape of a square surmounted by a semicircle. Its width is measured to be 40.5 inches and its area is computed. Find the approximate error in the computed area due to an error of $\frac{1}{8}$ of an inch in measuring the diameter.

CHAPTER VII

PERMUTATIONS, COMBINATIONS, AND PROBABILITY

78. Introduction. The formulas which will be used in this chapter depend on the following

THEOREM. *If an act which may be performed in p ways is followed by an act which may be performed in q ways, the total number of ways in which the two acts may be performed in succession is $p \cdot q$.*

For with each of the p ways of performing the first act one has a choice of q methods for the second. Hence with the entire p ways of performing the first there will be $p \cdot q$ ways of performing both acts.

For example, if there are 6 roads from A to B, and 4 from B to C, one has the choice of $6 \cdot 4 = 24$ routes in going from A to C through B.

EXERCISES

1. A room has 6 doors. In how many ways can a person enter and leave by a different door?

2. A man has 4 suits of clothes and 7 neckties. How many ways can he dress, not wearing the same tie twice with the same suit?

3. Two dice are thrown. In how many ways can they fall?

4. In presenting 8 men to 6 women how many introductions are made?

5. Three coins are pitched. In how many ways can they come up?

6. A town has 6 hotels. Three people wish to stay at different hotels. In how many ways can this be done?

7. In how many ways can two letters be posted in 4 letter boxes?

8. There are 25 stations on a branch line of a railroad. If both one-way and return tickets are sold between all stations, how many different kinds of tickets must be printed?

79. Permutations. Each different arrangement of a number of things is called a **permutation**. The letters A, B, and C may be arranged in the six different orders, ABC, ACB, BAC, BCA, CAB, CBA, each one of which is a permutation of the letters, distinct from the others. In determining how many permutations of these letters there are, we may employ the idea of successive acts as explained in the preceding section. Thus let the first act consist in filling the first place. This may be done with any one of the three letters, and hence in three different ways. With this place filled, there are only two letters left with which to fill the second place, which may be done in two ways. This affords $3 \cdot 2 = 6$ ways of filling the first two places. But with the first two places filled with two of the letters there is no choice in the way of filling the third place, as there is only one letter left. Hence the number of permutations of the three letters is 6.

THEOREM. *The number of permutations of n things taken r at a time is*

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

If only r of the n things are to be used at a time, there are only r places to be filled. Since the first place may be filled by any one of the n things, and the second place by any one of the $n-1$ remaining things, we see that the first two places may be filled in $n(n-1)$ ways. The third place may be filled by any one of the $n-2$ things which are left; hence the first three places may be filled in $n(n-1)(n-2)$ ways. Proceeding in this way, it appears that when $r-1$ places have been filled, we have left $n-(r-1)$ letters with which to fill the last place. Hence the rth place can be filled in $n-(r-1)$ ways, and $n-r+1$ is the last factor in the expression $_nP_r$.

COROLLARY. *The number of permutations of n things taken all at a time is*

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!.$$

The symbol $n!$ is read "factorial n." It is sometimes represented by $\underline{|n}$. In the foregoing theorem it is assumed that the elements are distinct, and that no element is used more than once in a given permutation.

EXAMPLE

If one has eight flags of different colors, how many signals can be displayed by showing them four at a time on a vertical line?

Solution. Here $n = 8$, $r = 4$. Hence $_8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$.

EXERCISES

1. How many arrangements of the letters in the word "Columbia" can be made, using in each arrangement (a) 4 letters? (b) all the letters?

2. Four people enter a room in which there are 7 vacant chairs. In how many ways can they be seated?

3. In how many orders can a hand of 6 cards be played?

4. With 5 flags of different colors, how many signals can be displayed by showing them any number at a time on a vertical line?

5. How many different numbers less than 1000 can be formed from the digits 1, 2, 3, 4, 5?

6. What is the number of permutations of the letters of the alphabet, taking three at a time?

7. How many numbers of 7 figures having 0 as middle digit can be formed from the digits 0, 1, 2, 3, 4, 5, 6?

8. In how many ways can 5 red books and 4 blue ones be arranged on a shelf so that all the books of each color are together?

80. Permutations with repetitions. Let us determine how many numbers of three digits can be written making use of the digits 2, 3, 4, and 5, where each digit may be used repeatedly. Here we have three places to fill. The first may be filled in any one of 4 ways, and, since repetition is allowed, the second and the third place may each be filled also in 4 ways. Hence all three places may be filled in $4 \cdot 4 \cdot 4 = 4^3$ ways. By similar reasoning we establish the following

THEOREM. *The numbers of permutations of n things taken r at a time when repetition is allowed is n^r.*

This theorem assumes that each thing may be repeated r times. If restriction is placed on the amount of repetition of one or more of the objects, the theorem is modified.

81. Permutations of things not all different. In the foregoing sections it has been assumed that all of the things to be permuted are different. If this is not the case, we have a modification of the formulas derived. In order to find the number of permutations of the letters in the word "algebra," taken all at a time, it is necessary to note that the letter a occurs twice. If for the moment these a's are considered as distinct, we shall have $7!$ permutations. But if in

each permutation the a's are treated as not distinct, we can interchange them without affecting the permutation; that is, the number of distinct permutations is $\frac{7!}{2}$.

THEOREM. *If, of n things, n_1 are alike, n_2 are alike but of another kind, n_3 are alike but of still another kind, etc., the number of distinct permutations of the n things, taken all at a time, is*

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots}.$$

For since the $n_1!$ permutations of the n_1 equal things are exactly alike, there will be only $\frac{1}{n_1!}$ times as many distinct permutations as there would be if these n_1 things were distinct. For a similar reason the total number, $n!$, of permutations of n things is divided by $n_2!$ because of the equality of the n_2 things, and so on.

When only a part of the n things are taken at a time, and some of them are alike, the situation is much more complicated and will not be considered here.

EXERCISES

1. How many different numbers less than 1000 can be formed from the digits 1, 2, 3, 4, 5, where each digit may be repeated?

2. Three dice are thrown. How many ways can they fall?

3. Find the number of distinct permutations of the letters of the word "mathematics," using all the letters in each permutation.

4. In how many ways can 4 coins be given to 10 boys, if each boy may receive any number of the coins?

5. Find the number of integers having 5 digits.

6. In how many ways can 6 letters be posted in 3 letter boxes?

7. (a) Find the number of distinct arrangements of the letters of the word "sophomore," using all the letters in each arrangement. (b) In how many of these arrangements do the 3 o's come together? (c) In how many of these arrangements do the 3 o's come at the end?

8. (a) Find the number of distinct arrangements of the letters of the word "engineering," using all the letters in each arrangement. (b) In how many of these arrangements will the i's not occur together?

82. Combinations. A group of objects which is independent of the order of its elements is called a **combination.** For example, a committee consisting of three men, A, B, and C, is the same committee whether we think of them as standing in the order ABC or CBA. It is evident, then, that there are more permutations of n things taken r at a time than there are combinations. The combination depends merely on the selection of the objects themselves and not at all upon the order in which they are arranged in the final groups. Since each combination of r things gives rise to $r!$ permutations, it appears that there are $r!$ times as many permutations of n things taken r at a time as there are combinations. This leads us to the

THEOREM. *The number of combinations of n things taken r at a time is*

$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{1}$$

This formula is easily remembered if one notices that there is the same number of factors in the numerator as in the denominator; that is, just r.

From the definition of combinations it is seen that the number of combinations of n things taken all at a time is 1.

THEOREM. *The number of combinations of n things taken r at a time is the same as the number of combinations of n things taken $n-r$ at a time.*

Expressed symbolically, $_nC_r = {_nC_{n-r}}$.

From (1), $$_nC_{n-r} = \frac{n(n-1)(n-2)\cdots[n-(n-r)+1]}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\cdots(r+1)}{(n-r)!}$$

multiplying numerator and denominator by $r!$,

$$= \frac{n(n-1)(n-2)\cdots(r+1)r(r-1)\cdots 2\cdot 1}{(n-r)!\ r!}$$

dividing numerator and denominator by $(n-r)!$,

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$= {_nC_r}.$$

The utility of this theorem will be appreciated if one compares the amount of computation involved in finding $_{21}C_3$ and its equal $_{21}C_{18}$.

For the solution of the exercises which follow, no specific rules can be laid down. In general, one should first observe whether the question involves combinations or permutations. If the latter, any possibility of repetition or equality of elements should be noted. So far as possible it is advisable to fall back on the principles on which the various formulas depend rather than to form the habit of using the formulas blindly.

EXERCISES

The first eight exercises involve only combinations.

1. Find the value of (a) ${}_7C_3$; (b) ${}_{50}C_{48}$.

2. How many committees of 9 can be selected from a group of 12 men?

3. How many crews of 8 men can be selected from a squad of 13 men?

4. How many straight lines are determined by (a) 7 points, no three of which are in the same straight line? (b) n points, no three of which are in the same straight line?

5. How many planes are determined by (a) 10 points, no four of which are in the same plane? (b) n points, no four of which are in the same plane?

6. Find n, if (a) ${}_nC_2 = 28$; (b) ${}_nC_3 = 84$.

7. Find n, if (a) ${}_nC_4 = {}_nC_2$; (b) ${}_nC_{n-3} = 35$.

8. How many different sums can be made up from a cent, a nickel, a dime, and a quarter?

9. If ${}_nP_r = 110$ and ${}_nC_r = 55$, find n and r.

10. If ${}_nC_3 = \frac{10}{21} \cdot {}_nC_5$, find n.

11. If ${}_nP_3 = 6\,{}_nC_4$, find n.

12. In how many ways can 7 coins be given to two boys so that one will get 3 and the other 4?

13. With 12 cadets, (a) in how many ways can a guard of 6 be chosen? (b) in how many ways can a guard of 6 be arranged in a line? (c) in how many ways can the 12 be divided into two equal groups?

14. A committee of 7 is to be chosen from 8 Englishmen and 5 Americans. In how many ways can the committee be chosen if it is to contain (a) just 4 Englishmen? (b) at least 4 Englishmen?

15. How many signals can be made by hoisting 8 flags, all at a time, on a staff, if 2 are white, 3 black, and the rest red?

16. How many signals can be made with the flags of exercise 15, using them all at a time, if a red flag is always at each extreme?

17. Show that the number of orders in which n things can be arranged in a circle is $(n-1)!$.

18. In how many orders can 7 men sit around a circular table?

19. In how many orders can 4 men and 4 women sit around a circular table so that a man is always between two women?

20. Out of 8 consonants and 3 vowels how many arrangements of letters, each containing 3 consonants and 2 vowels, can be formed?

21. How many handshakes may be exchanged among a party of 12 people, no two shaking hands more than once?

22. How many numbers greater than 100,000 can be formed by arranging the digits 1, 3, 0, 3, 2, 3?

23. The Greek alphabet has 24 letters. How many fraternity names can be formed, each containing three letters, repetition of letters being permitted?

24. In how many ways can a baseball team of 9 men be selected from 14 men, if only two of them can pitch, and these two can play in no other position?

25. How many telegraphic characters could be made by using 3 dots, 2 dashes, and 1 pause?

26. In how many ways may 15 passes and 5 failures be administered to a class of 20?

27. In how many ways can 7 men stand in line so that 2 particular men will not be together?

28. How many different sets of 4 hands can be dealt from a pack of 52 cards?

29. In how many ways may a football team of 11 men line up if the center and quarter back keep their positions, no line man being called back and no back being put in the line? It is assumed that in each line-up there are three men on each side of the center.

30. (a) How many diagonals has a decagon? (b) How many diagonals has a polygon of n sides?

31. Of 12 musicians 10 play the violin, 7 of these 10 also play the viola, and the remaining 5 play the cello. How many trios of different kinds of instruments can be made up?

32. How many triangles can be drawn, taking as vertices 8 points, just 3 of which lie in a straight line?

33. For a given value of n, what value of r affords the greatest value of ${}_nC_r$?

HINT. Since there are the same number of factors in the numerator and denominator of ${}_nC_r$, the smallest value of r which makes $\frac{n-r+1}{r} > 1$ will be the value sought.

83. Probability. If a bag contains 3 white balls and 4 black balls, and 1 ball is taken out at random, what is the chance that the ball drawn will be white?

This question we may answer as follows: There are 7 balls in the bag and we are as likely to get one as another. Thus a ball may be drawn in 7 different ways. Of these 7 possible ways 3 will produce a white ball. Thus the chance that the ball drawn will be white is 3 to 7, or $\frac{3}{7}$. The chance that a black ball will be drawn is $\frac{4}{7}$.

We may generalize this illustration as follows: If an event may happen in p ways and fail in q ways, each way being equally probable, the chance or probability that it will happen in one of the p ways is

$$\frac{p}{p+q}. \tag{1}$$

The chance that it will fail is

$$\frac{q}{p+q}. \tag{2}$$

The sum of the chances of the event's happening and failing is 1, as we see by adding (1) and (2).

The odds in favor of the event are the ratio of the chance of happening to the chance of failure. In this case the odds in favor are

$$\frac{p}{q}. \tag{3}$$

The odds against the event are $\frac{q}{p}$.

EXAMPLES

1. If the chance of an event's happening is $\frac{1}{10}$, what are the odds in its favor?

Solution. By (1), $$\frac{p}{p+q}=\frac{1}{10}.$$

Hence $$10p=p+q,$$

or $$9p=q,$$

or $$\frac{p}{q}=\frac{1}{9},$$ which by (3) are the odds in favor.

2. From a pack of 52 cards 3 are missing. What is the chance that they are all of a particular suit?

Solution. The number of combinations of 52 cards taken 3 at a time is ${}_{52}C_3=\frac{52\cdot 51\cdot 50}{1\cdot 2\cdot 3}$. This represents $p+q$. The number of combinations of the 13 cards of any one suit taken 3 at a time is ${}_{13}C_3=\frac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3}$. This represents p.

Thus $$\frac{p}{p+q}=\frac{\dfrac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3}}{\dfrac{52\cdot 51\cdot 50}{1\cdot 2\cdot 3}}=\frac{13\cdot 12\cdot 11}{52\cdot 51\cdot 50}=\frac{11}{17\cdot 50}=\frac{11}{850}.$$

3. What is the chance of throwing one and only one 6 in a single throw of 2 dice?

Solution. There are 36 possible ways for the two dice to fall. This represents $p+q$. Since a throw of two 6's is excluded, there are 5 throws in which each die would be a 6; that is, 10 in all in which a 6 appears. This represents p.

Thus $$\frac{p}{p+q}=\frac{10}{36}=\frac{5}{18}.$$

EXERCISES

1. A bag contains 8 white and 12 black balls. What is the chance that a ball drawn shall be (a) white? (b) black?

2. A bag contains 4 red, 8 black, and 12 white balls. What is the chance that a ball drawn shall be (a) red? (b) white? (c) not black?

3. In the previous problem, if 3 balls are drawn, what is the chance that (a) all are black? (b) 2 red and 1 white?

4. What is the chance of throwing neither a 3 nor a 4 in a single throw of 1 die?

5. What is the chance of throwing 7 in a single throw with 2 dice?

6. What is the chance of throwing three 5's in a single throw with 3 dice?

7. What is the chance of throwing 2 heads in a single throw with 2 coins? in 2 throws with 1 coin?

8. If 3 coins are thrown, what is the chance that just one will be a head?

9. Four men seated at a table match coins, agreeing that the odd man shall pay for the dinner. A remarks that it is likely to require several trials before one coin comes up different from all the others. B replies that the chances are even that this will happen on the first trial. Which is correct?

10. Three cards are drawn from a suit of 13. What is the chance that they will be ace, king, and queen?

11. Two cards are drawn from a pack of 52. What is the chance that they are both aces?

12. Four cards are drawn from a pack of 52. What is the chance that they are all clubs?

13. If 12 men stand in line, what is the chance that A and B are next to each other?

14. A man selects by lot 3 from a list of 10 friends to make up a dinner party. The list contains just 2 brothers. What is the chance that they are both invited?

15. If 3 dice are thrown, what are the odds in favor of at least 2 turning up alike?

16. Four men throw rackets for choosing partners in a game of tennis doubles. The 2 "smooths" and 2 "roughs" are to be partners. What are the odds against the choice being made on the first throw?

CHAPTER VIII

DETERMINANTS

84. Determinants of the second order. As a matter of notation it is agreed by mathematical writers to give the arrangement of letters $D_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ the meaning $a_1b_2 - a_2b_1$, where these letters may represent any numbers. The arrangement is called a **determinant.** Since there are two rows and two columns, the determinant is said to be of the second order. The expression $a_1b_2 - a_2b_1$ is called the **development** of the determinant. The value of the development of a determinant is often spoken of as the value of the determinant. The symbols a_1, b_1, a_2, b_2, are called **elements,** and a_1, b_2, are said to comprise the **principal diagonal** of the determinant.

Thus $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = 4 - 6 = -2$; $\begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix} = x\cdot 0 - y\cdot 1 = -y$.

The historical reason for this apparently artificial notation is the appearance of numbers in the form of the development $a_1b_2 - a_2b_1$ in the solution of a system of two linear equations in two variables.

Thus if we have given $$a_1x + b_1y = c_1 \tag{1}$$
and $$a_2x + b_2y = c_2, \tag{2}$$
we obtain by the usual method of solution,

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Using the determinant notation, we may write these results as follows:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

These expressions may be used as formulas for the solution of linear systems of equations in two variables.

The analogy between the solution of this simple system and the more complicated cases which follow will be seen more clearly if we observe that

I. *The determinants in the denominators are identical, and each consists of the coefficients of x and y as they stand in the original equations* (1) *and* (2).

II. *The determinant in the numerator of the value of x is formed from the denominator by replacing the coefficients of x, namely* $\begin{matrix} a_1 \\ a_2 \end{matrix}$, *by the constant terms* $\begin{matrix} c_1 \\ c_2 \end{matrix}$.

III. *The determinant in the numerator of the value of y is formed from the denominator by replacing the coefficients of y, namely* $\begin{matrix} b_1 \\ b_2 \end{matrix}$, *by the constant terms* $\begin{matrix} c_1 \\ c_2 \end{matrix}$.

85. Determinants of the third order. The arrangement of letters

$$D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

has been given the meaning

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2, \tag{2}$$

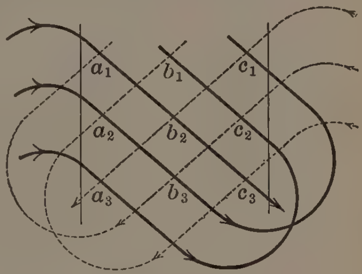
where the letters may take on any numerical values. The expression (1) is really an abbreviation or symbol for (2).

Since D_3 contains three rows and three columns, it is called a determinant of the third order, and (2) is called its development. The letters a_1, b_2, and c_3 constitute the principal diagonal. Similarly, we may have a determinant with n rows and n columns. This is called a determinant of the nth order.

Comparing this development with that of the determinant of the second order, we observe the following principle which will serve as a part of the rule for the development of determinants of orders higher than the third.

Each term of the development of a determinant of order 3 consists of a product of 3 elements, one from each row, and one from each column of the determinant.

This is verified in the case of (2) by observing that every one of the letters a, b, and c occurs in each term, and that every one of the subscripts 1, 2, and 3 also occurs in each of the terms of the development.



This statement gives us the law of formation for the development of a determinant of any order. The only feature which it does not cover is the determination of the signs of the terms. It so happens that for the determinant of the third order there is a simple rule for the determination of these signs. In the above figure the continuous lines indicate the right diagonals, while the dotted lines indicate the left diagonals. We may then state the

RULE. *To evaluate a determinant of the third order, multiply the numbers in each of the three right diagonals; multiply the numbers in each of the three left diagonals changing the sign of each product; then add the six products.*

It should be kept in mind that this rule does not apply to determinants of higher order than the third.

EXAMPLE

Evaluate $\begin{vmatrix} 3 & 2 & 1 \\ 4 & -6 & 2 \\ 1 & 0 & 1 \end{vmatrix}$.

$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & -6 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -18 + 0 + 4 + 6 - 0 - 8 = -16.$$

EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 4 & 1 & 6 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

5. $\begin{vmatrix} 5 & 3 & 7 \\ 15 & 9 & 21 \\ 3 & 1 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 0 \\ 6 & 1 & -2 \end{vmatrix}$.

4. $\begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix}$.

6. $\begin{vmatrix} 2 & -1 & 1 \\ 4 & 6 & -3 \\ 1 & 2 & 3 \end{vmatrix}$.

7. $\begin{vmatrix} a & x & y \\ 0 & b & c \\ 0 & c & b \end{vmatrix}$.

8. $\begin{vmatrix} a & b & c \\ -a & b & m \\ -a & -b & c \end{vmatrix}$.

9. $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ -1 & 0 \end{vmatrix}$.

10. $\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}$.

Solve the following equations:

11. $\begin{vmatrix} \sqrt{2}-x & x \\ x & \sqrt{2}+x \end{vmatrix} = 0.$

12. $\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} = 0.$

13. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & x & x^2 \\ 3 & 2 & 1 \end{vmatrix} = 0.$

14. $\begin{vmatrix} 1 & 1 & 1 \\ a & x & c \\ b & b & x \end{vmatrix} = 0.$

15. Show that

$$\begin{vmatrix} a^2 & h & g \\ h & b^2 & f \\ g & f & c^2 \end{vmatrix} = -(bg \pm ch)^2, \quad \text{if} \quad \begin{vmatrix} b^2 & f \\ f & c^2 \end{vmatrix} = 0.$$

86. Solution of linear equations in three variables. If we solve the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned} \qquad (3)$$

for x, y, and z, we obtain results which may be put in the form of determinants, in the following manner:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \qquad (4)$$

We could show by substitution that these values satisfy equations (3). It is customary to use them as formulas for the solution of a system of three linear equations in three variables. They may be remembered without difficulty if the following properties are verified from equations (4), and the analogy with the statements on page 152 is observed.

I. *The determinants in the denominators are identical, and each consists of the coefficients of x, y, and z, as they stand in the original equations.*

II. *Each determinant in the numerator is formed from the denominator by putting the column of constant terms (as they stand in the original equations) in place of the column of the coefficients of the variable whose value is sought.*

EXAMPLE

Solve the following equations by determinants:

$$\begin{aligned} x + y + z &= 2, \\ x + 3y - 4 &= 0, \\ y - 2z &= 6. \end{aligned}$$

Solution. Rearranging so that terms in the same variable are in the same column, and supplying the zero coefficients, we get

$$\begin{aligned} x + y + z &= 2, \\ x + 3y + 0z &= 4, \\ 0x + y - 2z &= 6. \end{aligned}$$

By (4), p. 154, $x = \dfrac{\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 0 \\ 6 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & -2 \end{vmatrix}} = \dfrac{-12 + 0 + 4 - 18 + 8 + 0}{-6 + 0 + 1 - 0 - 0 + 2} = \dfrac{-18}{-3} = 6,$

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 4 & 0 \\ 0 & 6 & -2 \end{vmatrix}}{-3} = \frac{-8 + 0 + 6 - 0 + 4 + 0}{-3} = \frac{2}{-3} = -\frac{2}{3},$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 1 & 6 \end{vmatrix}}{-3} = \frac{18 + 0 + 2 - 0 - 6 - 4}{-3} = \frac{10}{-3} = -\frac{10}{3}.$$

Check. Substituting the values found in the original equations, we have

$$\begin{aligned} 6 + (-\tfrac{2}{3}) + (-\tfrac{10}{3}) = 6 - \tfrac{12}{3} &= 6 - 4 = 2, \\ 6 + 3(-\tfrac{2}{3}) - 4 &= 6 - 2 - 4 = 0, \\ -\tfrac{2}{3} - 2(-\tfrac{10}{3}) &= -\tfrac{2}{3} + \tfrac{20}{3} = 6. \end{aligned}$$

EXERCISES

Solve the following systems of equations:

1. $x + y + z = 9,\quad x + 2y + 3z = 14,\quad x + 3y + 6z = 20.$

2. $2x + 3y = 12,\quad 3x + 2z = 11,\quad 3y + 4z = 10.$

3. $3x + y - z - 8 = 0,\quad 3y - 2x + z - 5 = 0,\quad x - y + 2z + 6 = 0.$

4. $2x + y = z,\quad 15x - 3y = 2\frac{1}{4}z,\quad 4x + 3y + 2z = 18.$

5. $\frac{1}{5}x - \frac{1}{2}y = 0,\quad \frac{1}{3}x - \frac{1}{2}z = 1,\quad \frac{1}{2}z - \frac{1}{3}y = 2.$

6. $3x + 4y - 2z = 5,\quad 4x - 3y + 8z = -4,\quad 2x + 8y - 3z - 5 = 0.$

7. $3x - 5y + 7z = 28,\quad 2x + 6y - 9z + 23 = 0,\quad 4x - 2y - 5z = 9.$

8. $x + y + z = 1,\quad \frac{x}{2} + \frac{y}{4} + 4z = 1,\quad \frac{5x}{3} + \frac{3y}{4} - \frac{z}{2} = 1.$

9. $x + y = z + a,\quad x + z = y + a,\quad y + z = x + a.$

10. $\frac{6}{x} + \frac{12}{y} - \frac{10}{z} = 4,\quad \frac{3}{x} + \frac{8}{y} + \frac{5}{z} = 4,\quad \frac{6}{x} + \frac{4}{y} + \frac{5}{z} = 4.$

87. Inversion. In order to find the development of a determinant with more than three rows and columns, the idea of an inversion is necessary. If in a series of positive integers a greater integer precedes a less, there is said to be an **inversion.** Thus in the series 1 2 3 4 there is no inversion, but in the series 1 2 4 3 there is one inversion, since 4 precedes 3. In 1 4 2 3 there are two inversions, as 4 precedes both 2 and 3; while in 1 4 3 2 there are three inversions, since 4 precedes 2 and 3, and also 3 precedes 2.

88. Development of the general determinant. We may write a determinant of the nth order as follows:

$$D_n = \begin{vmatrix} a_1 & b_1 & c_1 \cdots l_1 \\ a_2 & b_2 & c_2 \cdots l_2 \\ a_3 & b_3 & c_3 \cdots l_3 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n & b_n & c_n \cdots l_n \end{vmatrix}. \tag{1}$$

We can now define completely what is meant by the development of such a determinant.

Rule. *The development of a determinant of the nth order consists of the algebraic sum of all the terms which can be formed possessing the following properties:*

I. *Each term consists of the product of n elements, one from each row, and one from each column of the determinant.*

II. *The sign preceding each term is + or − according as the number of inversions of the subscripts of that term is even or odd, the order of the letters being the same as that of the principal diagonal.*

According to this rule, the sign of the term $a_3b_1c_2$ in the development of the determinant of the third order should be +, since the numbers 312 have two inversions. Reference to (2) of § 85 verifies this result. The signs of the other terms in (2) may be similarly obtained, and the use of the diagram on page 153 for the determination of the signs may be justified in this way.

An application of the first part of the preceding rule to a determinant of higher order than the third shows that the development of such a determinant contains more terms than would be obtained by taking the diagonals as explained for the determinant of the third order.

For example, if $n = 4$, the rule requires that $a_1b_2c_4d_3$ is a term of the development, although this set of elements does not occur in any diagonal of the determinant.

Since there are as many terms in the determinant of order n as there are arrangements of the subscripts $1, 2, \cdots, n$, it appears that the number of terms equals the number of permutations of n things taken all at a time, which is, by § 79, equal to $n!$.

For example, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ terms in the development of a determinant of the fifth order.

89. Properties of determinants. The meaning of the following theorems should be studied in the illustration with a determinant of the third order before the general proof is read.

I. *If each element of any row or column is multiplied by a constant, the value of the determinant is multiplied by that constant.*

Illustration.

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1b_2c_3 + ma_2b_3c_1 + ma_3b_1c_2 - ma_3b_2c_1 - ma_2b_1c_3 - ma_1b_3c_2$$
$$= m(a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3) = m \cdot D_3.$$

Proof. Since by the definition of the development of a determinant of the nth order given in the preceding section, each term must contain one and only one element from each row and each column, the factor m will appear just once in each term. If m is written outside a parenthesis, it appears that the parenthesis itself contains the development of the original D_n.

II. *The value of a determinant is not affected if the rows and the columns are interchanged.*

Illustration.

$$D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 \qquad (1)$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 = D_3.$$

Proof. If the rule for the development of the determinant D_n (§ 88) is applied to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \cdots a_n \\ b_1 & b_2 & b_3 \cdots b_n \\ c_1 & c_2 & c_3 \cdots c_n \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ l_1 & l_2 & l_3 \cdots l_n \end{vmatrix}$$

formed by interchanging the rows and columns of D_n, the two developments are identical. The signs of the corresponding terms are identical, since the principal diagonals in the two determinants are the same.

III. *The value of a determinant is changed in sign if two rows, or two columns, are interchanged.*

Illustration. Compare (1) with

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1 - a_2b_3c_1 - a_1b_2c_3 - a_3b_1c_2 = - D_3.$$

Proof. We need consider only the interchange of two rows, since, by II, rows and columns may be interchanged without affecting the value of the determinant.

An interchange of two adjacent rows does not affect the order of the letters in the principal diagonal or in any term of the development; it merely interchanges two adjacent subscripts in each term,

that is, it affords one more or one less inversion in the subscripts of each term, and hence changes the sign of each term. An interchange of adjacent rows or of adjacent columns is called a **transposition.**

An interchange of two rows separated from each other by m intermediate rows requires m transpositions to place the lower of the two rows next under the upper one, followed by $m+1$ transpositions to place the upper row in the place formerly occupied by the other, that is, $2m+1$ transpositions in all. But since each transposition causes a change of sign in the determinant, and since the whole process involves an odd number, $2m+1$, of such changes, we see that the determinant is left with a sign opposite to that of the original determinant.

IV. *If a determinant has two rows or two columns identical, the value of the determinant is zero.*

Proof. Let D_n be the value of the determinant, and let the two identical rows be interchanged. Then, by III, the sign of the determinant is changed. But since the rows which were interchanged were identical, D_n is really not affected at all; that is, $D_n = -D_n$, or $D_n = 0$.

V. *If each of the elements of any row or any column of a determinant consists of the sum of two numbers, the determinant may be expressed as the sum of two determinants.*

Illustration. Let the elements a_1, a_2, and a_3 in D_3 be replaced by $a_1' + a_1''$, $a_2' + a_2''$, and $a_3' + a_3''$ respectively. Then

$$\begin{aligned}
\begin{vmatrix} a_1' + a_1'' & b_1 & c_1 \\ a_2' + a_2'' & b_2 & c_2 \\ a_3' + a_3'' & b_3 & c_3 \end{vmatrix} &= (a_1' + a_1'')\, b_2c_3 + (a_3' + a_3'')\, b_1c_2 + (a_2' + a_2'')\, b_3c_1 \\
&\qquad - (a_3' + a_3'')\, b_2c_1 - (a_1' + a_1'')\, b_3c_2 - (a_2' + a_2'')\, b_1c_3 \\
&= a_1'b_2c_3 + a_3'b_1c_2 + a_2'b_3c_1 - a_3'b_2c_1 - a_1'b_3c_2 - a_2'b_1c_3 \\
&\qquad + a_1''b_2c_3 + a_3''b_1c_2 + a_2''b_3c_1 - a_3''b_2c_1 - a_1''b_3c_2 - a_2''b_1c_3 \\
&= \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.
\end{aligned}$$

Proof. In the determinant D_n, if each of the elements of any column, say the first, consists of the sum of two numbers, as $a_1' + a_1''$, $a_2' + a_2''$, $a_3' + a_3''$, $\cdots$, $a_n' + a_n''$, each term of the development will contain one of these binomials. If the parentheses containing these binomials in each term are removed, and the terms containing the a''s and the

a'''s are considered separately, it appears that we have the developments of two determinants whose first columns are the a''s and the a'''s respectively, and whose remaining columns are the same as those of the original determinant D_n.

VI. *If in a determinant the elements of a row or column are replaced by those elements plus the corresponding elements of another row or column, each multiplied by the same constant, the value of the determinant is unchanged.*

Illustration.

$$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = D_3 + m \cdot 0 = D_3.$$

The proof for the general case follows immediately from V, I, and IV.

90. Development by minors. The determinant which remains when the row and the column which contain a certain element are erased is called the **minor** of that element.

For example, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$; that of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

Since the development of a determinant contains only one element from any particular row, it appears that a certain number of the terms of the development contain, for example, a_1, but no other element of the first column; other terms contain a_2, still others a_3, and so on. In (2), § 85, for example, the first and last terms contain a_1, the second and the next to last contain a_2.

The essence of the method which follows consists in determining the exact nature of the coefficients of the elements of any row or column.

THEOREM. *The development of a determinant of order n may be written as the algebraic sum of n terms, each term consisting of two factors.*

The first factor of each term is an element of a certain row or column, each element being used but once.

The second factor of each term is the minor of the first factor of that term.

The element in the first factor is written with its sign changed, or unchanged, according as the sum of the number of the row and of the column in which it lies is odd or even.

Illustration.

$$\begin{vmatrix} -1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 3 & 4 \end{vmatrix} = -1\begin{vmatrix} 6 & -1 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= -(24+3) - 2(12+1) = -27 - 26 = -53;$$

or, developing with respect to the third column, we have

$$\begin{vmatrix} -1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 3 & 4 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} -1 & 2 \\ 3 & 6 \end{vmatrix} = (-3-2) + 4(-6-6)$$

$$= -5 - 48 = -53.$$

If any of the elements of a determinant are 0, it is shorter to develop the determinant with respect to a row or column having the most 0 elements.

Proof. Consider the coefficient of a_1 in the development of determinant (1) of § 88. The coefficient of a_1 consists of terms in each of which all of the other letters occur, and since all possible inversions of the subscripts $2, 3, \cdots, n$ are present, it must contain all of the terms of the minor of a_1. Since the removal of the subscript 1 from the first place in the series of subscripts $1, 2, \cdots, n$, leaves the same number of inversions in the subscripts $2, 3, \cdots, n$, that were originally present, it appears that each term of the coefficient of a_1 has the same sign as the corresponding term in the minor of a_1. Hence the coefficient of a_1 is precisely its minor.

Consider now any other element of the determinant, say the element in the kth row and lth column. By successive transpositions of rows and of columns we may bring this element into the leading position at the upper left corner. It requires $k-1$ transpositions of rows to bring the kth row to the top, and then $l-1$ transpositions of columns to bring the element in question into the leading position; that is, $k+l-2$ in all. Hence the sign of the determinant will have been changed $k+l-2$ times, and will be the same as it was originally if this number, or what amounts to the same thing, $k+l$, is even. If $k+l$ is odd, the sign of the determinant will have been changed by this process.

When the element in question is once in the leading position, its coefficient is its minor, by the reasoning which we went through with a_1; but if the sign of the determinant has been changed in getting it there, the term of the development which contains this element is changed in sign. Hence the theorem is established.

It is important to note that a given element has the same minor after a transformation of this kind that it had originally.

COROLLARY. *If in the development of a determinant by minors with respect to any row, the elements of this row are replaced by the elements of another row, the resulting expression vanishes.*

This follows from the fact that the expression which we obtain is virtually a determinant with two rows identical.

For example, if in the development of the determinant

$$D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3,$$

where A_1, A_2, and A_3 represent the minors of a_1, a_2, and a_3 respectively, we replace the a's by the b's, we have

$$b_1A_1 - b_2A_2 + b_3A_3,$$

which is equal to

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}.$$

Hence by IV, § 89,

$$b_1A_1 - b_2A_2 + b_3A_3 = 0.$$

91. Directions for evaluating determinants. In finding the value of a determinant with numerical elements it is frequently desirable to transform it in such a way that as many elements as possible in some row or column are zero, so that as many terms as possible in the development by minors will vanish. To this end the determinant should be scrutinized, with the following points in mind:

First, are there two rows which contain identical elements in like columns? If so, replace one of these rows by its own elements minus the corresponding elements of the other row.

Second, are several elements of any row each m times the corresponding elements of another row? If so, replace this row by its own elements minus m times the corresponding elements of the other row.

The mastery of these principles will carry with it the ability to use others in reducing numerical determinants to manageable form.

EXAMPLE

Evaluate the determinant

$$\begin{vmatrix} 35 & 65 & 25 & 12 \\ 17 & 26 & 12 & 5 \\ 4 & 5 & 2 & -7 \\ 3 & 5 & 2 & 1 \end{vmatrix}.$$

Solution. Subtracting 12 times the fourth row from the first, subtracting 5 times the fourth row from the second, and adding 7 times the fourth row to the third, we have

$$\begin{vmatrix} -1 & 5 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 25 & 40 & 16 & 0 \\ 3 & 5 & 2 & 1 \end{vmatrix}.$$

Now developing by minors with respect to the fourth column, the element 1, being in the fourth row and fourth column, retains its sign $+$, since $4+4$ is an even number, and the determinant reduces to one of the third order

$$\begin{vmatrix} -1 & 5 & 1 \\ 2 & 1 & 2 \\ 25 & 40 & 16 \end{vmatrix}.$$

Now subtract twice the first row from the second and 16 times the first row from the third; this gives

$$\begin{vmatrix} -1 & 5 & 1 \\ 4 & -9 & 0 \\ 41 & -40 & 0 \end{vmatrix} = \begin{vmatrix} 4 & -9 \\ 41 & -40 \end{vmatrix} = -160 + 369 = 209.$$

It should be noted that a row whose multiple is combined with another is left unchanged.

EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 13 & 17 & 4 \\ 28 & 33 & 8 \\ 40 & 54 & 13 \end{vmatrix}.$

2. $\begin{vmatrix} k & a & b+c \\ k & b & c+a \\ k & c & a+b \end{vmatrix}.$

3. $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$

4. $\begin{vmatrix} 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 3 \\ 3 & 1 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{vmatrix}.$

5. $\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}$.

6. $\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$.

7. $\begin{vmatrix} 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2 & 3 & a \\ 2 & 3 & 4 & b \\ 3 & 4 & 5 & c \\ 4 & 5 & 6 & d \end{vmatrix}$.

9. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$.

10. $\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix}$.

11. $\begin{vmatrix} 47 & 5 & 2 & 91 \\ 54 & 6 & 3 & 92 \\ 28 & 3 & 3 & 93 \\ 0 & 0 & 0 & 5 \end{vmatrix}$.

12. $\begin{vmatrix} 11 & 12 & 8 & 1 \\ 10 & 17 & 21 & 3 \\ 15 & 38 & 19 & 2 \\ 6 & 11 & 8 & 1 \end{vmatrix}$.

13. $\begin{vmatrix} x+1 & 1 & 1 & 1 \\ 1 & y+1 & 1 & 1 \\ 1 & 1 & z+1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$.

14. $\begin{vmatrix} a & x & y & z \\ x_1 & b & 0 & 0 \\ y_1 & 0 & c & 0 \\ z_1 & 0 & 0 & d \end{vmatrix}$.

15. $\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ -2 & 3 & 2 & 2 & 2 \\ -3 & 2 & 3 & 2 & 2 \\ -2 & 2 & 2 & 3 & 2 \\ -2 & 2 & 2 & 2 & 3 \end{vmatrix}$.

16. $\begin{vmatrix} a & 0 & b & 0 & x \\ c & 0 & d & x & e \\ f & 0 & x & 0 & 0 \\ g & x & h & i & j \\ x & 0 & 0 & 0 & 0 \end{vmatrix}$.

17. Show that $\begin{vmatrix} x^2 & xy & y^2 \\ 2x & x+y & 2y \\ 1 & 1 & 1 \end{vmatrix} = (x-y)^3$.

18. Show that if all the elements of a determinant of order n which lie on one side of the principal diagonal are zero, the value of the determinant is equal to the product of the elements of the principal diagonal.

Solve the following systems of equations:

19. $$\begin{aligned} ax + by + cz &= a, \\ bx + cy + az &= b, \\ cx + ay + bz &= c. \end{aligned}$$

20. $$\begin{aligned} x + y + z &= a + b + c, \\ bx + cy + az &= ab + bc + ca, \\ cx + ay + bz &= ab + bc + ca. \end{aligned}$$

21. $$\begin{aligned} x + y &= z + 2c, \\ x + z &= y + 2b, \\ y + z &= x + 2a. \end{aligned}$$

22. $$\begin{aligned} ax + by - cz &= 2ab, \\ by + cz - ax &= 2bc, \\ cz + ax - by &= 2ac. \end{aligned}$$

92. Solution of systems of linear equations. Suppose that we have given n linear equations in n variables. We seek a solution of the equations in terms of determinants. For simplicity let $n = 4$. Given

$$a_1x + b_1y + c_1z + d_1w = f_1, \tag{1}$$

$$a_2x + b_2y + c_2z + d_2w = f_2, \tag{2}$$

$$a_3x + b_3y + c_3z + d_3w = f_3, \tag{3}$$

$$a_4x + b_4y + c_4z + d_4w = f_4. \tag{4}$$

The coefficients of the variables in the order in which they are written may be taken as forming a determinant D_4, which we call the **determinant of the system**. We assume that $D_4 \neq 0$. Thus

$$D_4 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Symbolize by A_1, B_3, etc., the minors of a_1, b_3, etc., in this determinant. Let us solve for x. Multiply (1), (2), (3), (4) by A_1, $-A_2$, A_3, $-A_4$ respectively. We obtain

$$\begin{aligned} A_1a_1x + A_1b_1y + A_1c_1z + A_1d_1w &= A_1f_1, \\ -A_2a_2x - A_2b_2y - A_2c_2z - A_2d_2w &= -A_2f_2, \\ A_3a_3x + A_3b_3y + A_3c_3z + A_3d_3w &= A_3f_3, \\ -A_4a_4x - A_4b_4y - A_4c_4z - A_4d_4w &= -A_4f_4. \end{aligned}$$

If we add these equations, the coefficient of x is the determinant D_4, while the coefficients of y, z, w, are zero (by the corollary of § 90).

The right member of the equation is the determinant D_4, except that the elements of the first column are replaced by f_1, f_2, f_3, f_4 respectively. Hence

$$x = \frac{\begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ f_3 & b_3 & c_3 & d_3 \\ f_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}.$$

In a similar manner we can show that the value of any variable which satisfies the equations is given by the following

RULE. *The value of one of the variables in n linear equations in n variables consists of a fraction whose denominator is the determinant of the system and whose numerator is the same determinant, except that the column which contains the coefficients of the given variable is replaced by the column consisting of the constant terms.*

When the determinant of the system is zero, we cannot solve the equations without further discussion, which may be found in more advanced treatises.

EXAMPLE

Solve for x the following system of equations:

$$ax + 2by = 1,$$
$$2by + 3cz = 2,$$
$$3cz + 4dw = 3,$$
$$4dw + 5ax = 4.$$

Solution. Rearranging, we obtain

$$\begin{array}{llllll} ax + 2by & & & & = 1, \\ & 2by + 3cz & & & = 2, \\ & & 3cz + 4dw & & = 3, \\ 5ax & & & + 4dw & = 4. \end{array}$$

$$x=\frac{\begin{vmatrix}1 & 2b & 0 & 0\\ 2 & 2b & 3c & 0\\ 3 & 0 & 3c & 4d\\ 4 & 0 & 0 & 4d\end{vmatrix}}{\begin{vmatrix}a & 2b & 0 & 0\\ 0 & 2b & 3c & 0\\ 0 & 0 & 3c & 4d\\ 5a & 0 & 0 & 4d\end{vmatrix}}=\frac{24\,bcd\begin{vmatrix}1 & 1 & 0 & 0\\ 2 & 1 & 1 & 0\\ 3 & 0 & 1 & 1\\ 4 & 0 & 0 & 1\end{vmatrix}}{24\,abcd\begin{vmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 5 & 0 & 0 & 1\end{vmatrix}}=\frac{\begin{vmatrix}1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 3 & 0 & 1 & 1\\ 4 & 0 & 0 & 1\end{vmatrix}}{a\begin{vmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & -5 & 0 & 1\end{vmatrix}}$$

$$=\frac{-\begin{vmatrix}1 & 1 & 0\\ 3 & 1 & 1\\ 4 & 0 & 1\end{vmatrix}}{a\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ -5 & 0 & 1\end{vmatrix}}=\frac{-\begin{vmatrix}1 & 1 & 0\\ 2 & 0 & 1\\ 4 & 0 & 1\end{vmatrix}}{a\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 5 & 1\end{vmatrix}}=\frac{\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix}}{a\begin{vmatrix}1 & 1\\ 5 & 1\end{vmatrix}}=\frac{-2}{-4a}=\frac{1}{2a}.$$

93. Solution of homogeneous linear equations. A **homogeneous equation** is one in which all the terms are of the same degree in the variables.

The equations considered in the previous section become homogeneous if $f_1=f_2=f_3=f_4=0$. We have then

$$\begin{aligned} a_1x+b_1y+c_1z+d_1w&=0,\\ a_2x+b_2y+c_2z+d_2w&=0,\\ a_3x+b_3y+c_3z+d_3w&=0,\\ a_4x+b_4y+c_4z+d_4w&=0. \end{aligned} \tag{1}$$

These equations are evidently satisfied by $x=y=z=w=0$. This we call the zero solution. We seek the condition which the coefficients must fulfill in order that other solutions also may exist. If we carry out the method of the previous section, we observe that the determinant in the numerator of every fraction which affords the value of one of the variables equals zero. Thus if D_4 is not equal to zero, the only solution of the above equations is the zero solution. This gives us the following

PRINCIPLE. *A system of n linear homogeneous equations in n variables has a solution distinct from the zero solution only when the determinant of the system vanishes.*

Whether a solution distinct from the zero solution *always* exists when the determinant of the system equals zero, we shall not determine, as a complete discussion of the question is beyond the scope of this chapter.

In any particular case a system of linear equations whose determinant has been found to equal zero may usually be solved by dividing all of the equations by one of the variables, as w in (1), and solving the first $n-1$ equations for the ratios $\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, \cdots$. When values of these ratios can be found, they will always satisfy the remaining equation, and in this way an infinite number of sets of roots may be obtained.

EXAMPLE

Solve the following system of homogeneous equations:

$$5x + 4y + z = 0,$$
$$5x + 3y + 2z = 0,$$
$$6x + y + 5z = 0.$$

Solution. Here the determinant of the system

$$D = \begin{vmatrix} 5 & 4 & 1 \\ 5 & 3 & 2 \\ 6 & 1 & 5 \end{vmatrix} = 0.$$

Dividing each of the equations by z, we have

$$5\frac{x}{z} + 4\frac{y}{z} = -1,$$
$$5\frac{x}{z} + 3\frac{y}{z} = -2,$$
$$6\frac{x}{z} + \frac{y}{z} = -5.$$

Solving the first two of these equations for $\frac{x}{z}$ and $\frac{y}{z}$,

$$\frac{x}{z} = -1, \quad \frac{y}{z} = 1.$$

These values satisfy the third equation. Hence x, y, and z may be any numbers which satisfy the relation

$$x : y : z = 1 : -1 : -1,$$

or

$$x = -y = -z.$$

Thus the system is satisfied by the sets of numbers,

$$1, -1, -1; \quad 2, -2, -2, \text{ etc.}$$

EXERCISES

Solve the following systems of equations:

1. $$\begin{aligned} -w+x+y+z&=8,\\ w-x+y+z&=6,\\ w+x-y+z&=4,\\ w+x+y-z&=2. \end{aligned}$$

2. $$\begin{aligned} v+3w-11&=0,\\ w+3x-15&=0,\\ x+3y-19&=0,\\ y+3z-8&=0,\\ z+3v-7&=0. \end{aligned}$$

3. $$\begin{aligned} x+3y-z&=1,\\ y+3z-w&=4,\\ z+3w-x&=11,\\ w+3x-y&=2. \end{aligned}$$

4. $$\begin{aligned} w+x-z&=2,\\ x+2y-3w&=4,\\ 3x-5y+2z&=-1,\\ 2w+y-z&=0. \end{aligned}$$

Solve the following systems of homogeneous equations:

5. $$\begin{aligned} 3x+5y+6z&=0,\\ 2x+4y+5z&=0,\\ 4x+6y+7z&=0. \end{aligned}$$

6. $$\begin{aligned} x+2y+3z&=0,\\ 2x+3y+4z&=0,\\ 3x+4y+5z&=0. \end{aligned}$$

7. $$\begin{aligned} 2x+3y+2z&=0,\\ 3x+y+4z&=0,\\ 4x-2y-z&=0. \end{aligned}$$

8. $$\begin{aligned} 6x+y-7z&=0,\\ 5x-10y+5z&=0,\\ 4x+3y-7z&=0. \end{aligned}$$

CHAPTER IX

PARTIAL FRACTIONS

94. Introduction. In the integral calculus it is often necessary to express a fraction in which the denominator is an integral rational function of one variable, as the sum of several fractions each of which has a linear, or at most a quadratic, function in the denominator. It is always assumed in what follows that the denominator of the original expression is of higher degree in x than the numerator. Whenever this is not the case, it is necessary to reduce the fraction by long division to a mixed expression in which the fractional part is in the desired form. This step, which is preliminary to all of the cases, we state as follows:

If in $\dfrac{f(x)}{\phi(x)}$, $f(x)$ *is of higher degree than* $\phi(x)$, *express by means of long division the fraction in the form* $\dfrac{f(x)}{\phi(x)} = Q(x) + \dfrac{R(x)}{\phi(x)}$, *where* $R(x)$, *the remainder in the division, is of lower degree than* $\phi(x)$.

95. Denominator with distinct linear factors. The essence of the following method consists in assuming that the fraction can be expressed in the following form, and then seeking to determine the numerators which in the assumed form are left undetermined.

EXAMPLE

Separate into partial fractions

$$\frac{x+4}{(x-1)(x-2)(x-3)}.$$

Solution. Assume

$$\frac{x+4}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{1}$$

To determine what numerical values A, B, and C must have, multiply both sides of the equation by $(x-1)(x-2)(x-3)$.

We have $x + 4 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$.

Since this expression is assumed to be an identity, it must be true for all values of x (§ 11). If, then, we let x take on the value 1, all of the terms in the right member vanish excepting the one containing A. We can then find the value of A.
$$1+4=A(1-2)(1-3), \text{ or } A=\tfrac{5}{2}.$$

Similarly, we find the values of B and C by letting x take on the values 2 and 3 respectively. Thus $B=-6$, $C=\frac{7}{2}$.

Hence
$$\frac{x+4}{(x-1)(x-2)(x-3)}=\frac{5}{2(x-1)}-\frac{6}{x-2}+\frac{7}{2(x-3)}.$$

In this text we shall not prove that the assumption analogous to (1) always leads to the partial fractions which we seek. In any particular case the fractions obtained should be added as a check on the work, and the validity of the result in this way determined.

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2-2}{x(x-1)(x-2)}$.

2. $\dfrac{x^2+4}{(x-2)(x+2)(x-1)}$.

3. $\dfrac{x}{x^2+11x+30}$.

4. $\dfrac{3+2x}{(2x-3)(5x-4)}$.

5. $\dfrac{2x^2-1}{(x-1)(x^2+3x+2)}$.

6. $\dfrac{x}{(x+1)(x+3)(x+5)}$.

96. Denominator with distinct linear and quadratic factors. The method of treating this case is similar to that shown in the preceding section. The assumption is slightly different.

The numerator of a partial fraction with a linear denominator is always assumed to be a constant. The numerator of a partial fraction with a prime quadratic factor (see p. 1) in the denominator is always assumed to be linear and of the form $Ax+B$.

EXAMPLE

Separate into partial fractions:
$$\frac{5x^2+8x+11}{(x^2+1)(x-3)(x+1)}.$$

Solution. In this example we make the assumption
$$\frac{5x^2+8x+11}{(x^2+1)(x-3)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-3}+\frac{D}{x+1}.$$

Multiplying by $(x^2+1)(x-3)(x+1)$, we have $5x^2+8x+11=Ax(x-3)(x+1)+B(x-3)(x+1)+C(x^2+1)(x+1)+D(x^2+1)(x-3)$.

Letting $x = -1$, we obtain $D = -1$; letting $x = 3$, we obtain $C = 2$. Substituting these values for C and D, and letting $x = 0$, we obtain

$$B = -2.$$

Substituting the values already found, and letting $x = 1$, we obtain

$$A = -1.$$

Hence $$\frac{5x^2 + 8x + 11}{(x^2+1)(x-3)(x+1)} = \frac{-x-2}{x^2+1} + \frac{2}{x-3} - \frac{1}{x+1}$$

EXERCISES

Separate into partial fractions:

1. $\dfrac{1 + x + x^2}{x(x^2+4)}$.
2. $\dfrac{x^2 + 15}{(x-1)(x^2+2x+5)}$.
3. $\dfrac{3 - x^2}{2 + x + 2x^2 + x^3}$.
4. $\dfrac{x}{(x+3)(2x^2-x-4)}$.
5. $\dfrac{3x + 4}{(x^2-x)(x^2+1)}$.
6. $\dfrac{x^3 + 5}{x^4 + x^3 + x^2 + x}$.

97. Denominator with repeated factors. If the fraction is in the form $\dfrac{f(x)}{(x-a)^n}$, replace x by $y + a$ in both numerator and denominator, and simplify the numerator. The partial fractions are directly obtained, and may be expressed in terms of x by replacing y by $x - a$.

EXAMPLE

Separate into partial fractions:

$$\frac{3x^2 - 4x + 3}{(x-2)^3}.$$

Solution. Letting $x = y + 2$,

$$\frac{3x^2 - 4x + 3}{(x-2)^3} = \frac{3(y+2)^2 - 4(y+2) + 3}{(y+2-2)^3} = \frac{3y^2 + 8y + 7}{y^3}$$

$$= \frac{3}{y} + \frac{8}{y^2} + \frac{7}{y^3} = \frac{3}{x-2} + \frac{8}{(x-2)^2} + \frac{7}{(x-2)^3}.$$

When a factor of the form $(x-a)^k$ appears together with other factors in the denominator of a fraction which is to be broken up into partial fractions, the assumed form is taken as in the following example, and the coefficients may be found by replacing x by convenient integers. It is frequently impossible to find integers which will enable us to determine a coefficient at each substitution, but systems of equations are obtained from which the coefficients may be found.

EXAMPLE

Separate into partial fractions:

$$\frac{5x^2-6x-5}{(x-1)^3(x+2)}.$$

Solution. Assume

$$\frac{5x^2-6x-5}{(x-1)^3(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+2}.$$

Multiplying by $(x-1)^3(x+2)$,

$$5x^2-6x-5=A(x-1)^2(x+2)+B(x-1)(x+2)+C(x+2)+D(x-1)^3.$$

Letting $x=-2$, we find $D=-1$; letting $x=1$, we find $C=-2$.

Substituting these values of C and D, and letting $x=0$, we obtain

$$A-B=-1.$$

Letting $x=2$, we obtain $A+B=3.$

Solving, $A=1,\ B=2.$

Hence
$$\frac{5x^2-6x-5}{(x-1)^3(x+2)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{2}{(x-1)^3}-\frac{1}{x+2}.$$

EXERCISES

1. $\dfrac{x}{(x-4)^3}.$

2. $\dfrac{3+x}{(5-x)^2}.$

3. $\dfrac{2x+5}{(x-3)(x-1)^3}.$

4. $\dfrac{2x^2-6x-5}{(2x+3)^3}.$

5. $\dfrac{2x^4-1}{x^4(x+1)}.$

6. $\dfrac{x^4-6x^3+4x^2-2x+3}{x^3(x^2+1)}.$

98. General directions. The following statements indicate the form of the assumed partial fractions in each case, corresponding to the types of factors in the denominator of the original fraction.

I. *Corresponding to the factor $x-a$, assume the partial fraction*

$$\frac{A}{x-a}.$$

II. *Corresponding to the prime factor ax^2+bx+c, assume the partial fraction*

$$\frac{Ax+B}{ax^2+bx+c}.$$

III. *Corresponding to the factor* $(x-a)^k$, *assume the sum*

$$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\cdots+\frac{K}{(x-a)^k}.$$

IV. *Corresponding to the factor* $(ax^2+bx+c)^k$, *assume the sum*

$$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\cdots+\frac{Mx+N}{(ax^2+bx+c)^k}.$$

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2+x+1}{x^4+4x^2}$.

2. $\dfrac{x^2+1}{(x+1)^3}$.

3. $\dfrac{x^3+1}{x(x-1)^3}$.

4. $\dfrac{x^2+x+1}{x^3-4x^2+x+6}$.

5. $\dfrac{2x^3+3x^2+x}{(x^2+x+1)^2}$.

6. $\dfrac{x^3}{(x-1)^2(x^2+2)}$.

7. $\dfrac{x}{x^3-a^3}$.

8. $\dfrac{a^2-x^2+2amx}{(am-x)(a^2+x^2)}$.

9. $\dfrac{2x^3+5x^2-6}{x^4+2x^3-x^2-2x}$.

10. $\dfrac{1}{x(x^2+1)^2}$.

11. $\dfrac{2v^2-1}{3v^3-3v}$.

12. $\dfrac{y^2+y+1}{(a+1+y)[a(y+1)+y]}$.

13. $\dfrac{x^3-x^2-1}{(x-1)^2}$.

14. $\dfrac{x^4+x^2}{x^4+x^2+1}$.

15. $\dfrac{x^2+x}{(x-1)^2(x^2+4)}$.

16. $\dfrac{4x^3-5x+2}{x^6-x^5-x^4+x^3}$.

17. $\dfrac{3x^3+x^2+x+1}{(5x^2-x+1)^2}$.

18. $\dfrac{2}{3(2x+1)(x^2+x+3)}$.

19. $\dfrac{x^2+1}{(x^2-1)(x^2-5x+6)}$.

20. $\dfrac{2+x^3}{(2-x)^2(1+x^2)}$.

CHAPTER X

LOGARITHMS

99. Introduction. Up to the present point in this book only rational numbers have been used as exponents. The fractional exponent was assumed in § 2 to obey the laws of operation which govern the integral exponents, and its meaning was defined. But the meaning of a number with an irrational exponent, like $3^{\sqrt{2}}$, has not as yet been considered. To treat this subject and all of the delicate questions which are connected with it is at present impracticable, but the general meaning of an irrational exponent is not difficult to understand. Every irrational number is capable of approximate expression in terms of decimals, and can be computed to as many places as necessary by the application of some numerical law. For example, $\sqrt{2}$ may be found to as many decimal places as we desire by the ordinary process of extracting the square root. None of the approximations 1.4, 1.41, 1.414, 1.4142, is exactly $\sqrt{2}$, but each one is correct to as many places of decimals as it contains, and consequently each number differs by less than the preceding one from the exact value of $\sqrt{2}$. The value of a number with an irrational exponent may be approximated in a similar manner. For example, $3^{\sqrt{2}}$ may be computed with increasing accuracy by allowing $\sqrt{2}$ to take on a succession of approximations like that given above. The value of $3^{1.414} = 3^{\frac{1414}{1000}} = \sqrt[1000]{3^{1414}}$ could be computed directly only with difficulty. The simplest method which has been explained would be the solution of the equation $x^{1000} = 3^{1414}$ by Horner's Method.

In what follows we shall assume that exponents may be irrational numbers, and that the formal laws of operation with these exponents are the same as already given on page 4 for the case of rational exponents.

100. Definition of logarithm. In the preceding section the possibility of finding a in the expression

$$a = b^x, \tag{1}$$

when b and x are given, was discussed. The reverse process of finding x when a and b are given gives rise to logarithms.

DEFINITION. *The logarithm of a given number is the exponent in the power to which a number, called the base, must be raised in order to equal the given number.*

In (1) x is the logarithm of a for the base b, or, symbolically expressed, $x = \log_b a$. Thus

$$b^x = a \quad \text{and} \quad x = \log_b a \tag{2}$$

are equivalent relations. The foregoing definition assumes that when a and b are given, the real number x always exists, an assumption which is justified when both a and b are positive and $b \neq 1$.

Although, theoretically, any positive number excepting unity could be taken for the base of a system of logarithms, the only ones which are ever used in computations are 10, and the number $e = 2.7128 \cdots$ (the Napierian base), which we shall meet later. When the base is not expressed, as in log 2, the base 10 is understood, since 10 is the usual base for purposes of computation.

EXERCISES

1. If $2^x = 16$, what is the value of x? $\log_2 16 = ?$

2. Find the values of $\log_3 27$, $\log_{10} 1000$, $\log_{\frac{1}{2}} \frac{1}{4}$, $\log_{\sqrt{7}} 49$.

3. If $b^3 = 343$, what is the value of b? If $\log_b 343 = 3$, $b = ?$

4. Find the base b in each of the following: $\log_b 32 = 5$; $\log_b 100 = 2$; $\log_b .008 = 3$.

5. If $4^2 = a$, what is the value of a? If $\log_4 a = 2$, $a = ?$

6. Find the number a in each of the following: $\log_6 a = 2$; $\log_{10} a = 4$; $\log_{.5} a = 3$.

7. Find the value of $\log_3 1$; $\log_{23} 1$. Show that $\log_b 1 = 0$ for any positive base b; that is, in any system of logarithms $\log 1 = 0$.

8. What is the value of $\log_1 1$? $\log_1 2$? $\log_1 a$? Why can 1 not be used as base for a system of logarithms?

9. What is the value of $\log_{-2} 4$? $\log_{-2} 8$? Why can a negative number not be used as base for a system of logarithms?

10. What is the value of $\log_2 (-4)$? $\log_{-2} (-8)$? $\log_b (-a)$? Show that negative numbers have no real logarithms for positive bases.

11. What is the value of log 100? log 10? log 1? log .1? log .01? Which numbers have positive and which have negative logarithms?

12. What is the value of $\log_2 2$? $\log_5 5$? $\log_a a$? Show that in any system the logarithm of the base is 1.

13. If $\sqrt{a} = \sqrt{b} \cdot b^x$, find $\log_b a$.

14. If $a^2 = 100\sqrt[3]{10}$, find $\log a$.

15. If $ae^{-3} = (e^t)^{\frac{2}{3}} e^{\frac{8}{3}}$, find $\log_e a$.

16. Show that $\log_b a = \dfrac{1}{\log_a b}$.

17. Show that $b^{\log_b a} = a$.

18. Show that $\log_{\frac{1}{b}} a = \log_b \dfrac{1}{a} = -\log_b a$.

19. Show that $\log_b a \cdot \log_c b \cdot \log_a c = 1$.

101. Operations with logarithms. The fact that **a logarithm is an exponent** lies at the basis of its usefulness, since it enables us to employ the laws of exponents with telling effect. We now prove four theorems which enable us to apply logarithms in numerical computations.

THEOREM I. *The logarithm of the product of two numbers is the sum of their logarithms.*

Symbolically expressed, $\log_b (ac) = \log_b a + \log_b c$.

Let
$$\log_b a = x,$$
$$\log_b c = y.$$

Then by (2), p. 176,
$$b^x = a,$$
$$b^y = c.$$

Multiplying,
$$b^{x+y} = a \cdot c,$$

or by (2)
$$\log_b (ac) = x + y = \log_b a + \log_b c.$$

THEOREM II. *The logarithm of the nth power of a number is n times the logarithm of the number.*

Symbolically expressed, $\log_b a^n = n \log_b a$.

Let
$$\log_b a = x,$$

or
$$b^x = a.$$

Raising both sides to the nth power,
$$(b^x)^n = b^{nx} = a^n,$$

or
$$\log_b a^n = nx = n \log_b a.$$

THEOREM III. *The logarithm of the quotient of two numbers is the logarithm of the numerator minus the logarithm of the denominator.*

Symbolically expressed, $\log_b \frac{a}{c} = \log_b a - \log_b c.$

Let $\log_b a = x,$

$\log_b c = y.$

Then $b^x = a,$

$b^y = c.$

Dividing, $b^{x-y} = \frac{a}{c},$

or $\log_b \frac{a}{c} = x - y = \log_b a - \log_b c.$

THEOREM IV. *The logarithm of the real nth root of a number is the logarithm of the number divided by n.*

Symbolically expressed, $\log_b \sqrt[n]{a} = \frac{\log_b a}{n}.$

Let $\log_b a = x,$

or $b^x = a.$

Extracting the nth root, $(b^x)^{\frac{1}{n}} = b^{\frac{x}{n}} = \sqrt[n]{a},$

or $\log_b \sqrt[n]{a} = \frac{x}{n} = \frac{\log_b a}{n}.$

EXAMPLE

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$. Find $\log(\sqrt[5]{7^3} \cdot \sqrt{5})$.

Solution.

$$\log(\sqrt[5]{7^3} \cdot \sqrt{5}) = \tfrac{3}{5}\log 7 + \tfrac{1}{2}\log 5,$$
$$\tfrac{3}{5}\log 7 = \tfrac{3}{5}(.8451) = .5071,$$
$$\tfrac{1}{2}\log 5 = \tfrac{1}{2}(.6990) = .3495,$$
$$\log(\sqrt[5]{7^3} \cdot \sqrt{5}) = \overline{.8566}.$$

NOTE. In using four places of decimals, when the number in the fifth place is less than 5 it is dropped; when it is more than 5, 1 is added to the number in the fourth place. When it is exactly 5, 1 is added to the number in the fourth place in case that number is odd; otherwise it is dropped. Thus $\frac{3}{5}\log 7$ came out .50706, which, to four places, is .5071.

EXERCISES

Using the logarithms on the preceding page, find:

1. $\log 210$.
2. $\log 32$.
3. $\log 18$.
4. $\log 1225$.
5. $\log 2\sqrt[3]{7}$.
6. $\log 9\sqrt{10}$.
7. $\log(\sqrt[5]{3}\cdot\sqrt[3]{7^4})$.
8. $\log(\sqrt[7]{14}\cdot\sqrt[14]{7})$.
9. $\log\dfrac{3\sqrt{15}}{7}$.
10. $\log\dfrac{81}{(\sqrt[3]{7})^2}$.
11. $\log\dfrac{8\sqrt[3]{2}\cdot 9^{10}}{7^5\cdot 5^7}$.
12. $\log\sqrt{\dfrac{9^3\cdot\sqrt{2}}{105}}$.

13. If the edge of a cube is a, its surface S, and its volume V, show that (a) $\log S = 2\log a + .7781$; (b) $\log V = 3\log a$.

14. If the edge of a regular tetrahedron is a, its surface S, and its volume V, show that

(a) $\log S = 2\log a + .2386$; (b) $\log V = 3\log a - .8286$.

Deduce the following relations:

15. $\log\left(b - \dfrac{a^2}{b}\right) = \log(b+a) + \log(b-a) - \log b$.

16. $\log\left(1 - \dfrac{2b}{a} + \dfrac{b^2}{a^2}\right)^3 = 6[\log(a-b) - \log a]$.

17. $\log\left[\dfrac{1}{2(1-x)} + \dfrac{1}{2(1+x)}\right] = -[\log(1-x) + \log(1+x)]$.

18. $\log\sqrt{(2x-3)(x-1)-6} = \frac{1}{2}[\log(2x+1) + \log(x-3)]$.

19. $\log\sqrt[3]{\dfrac{(x+1)^3+1}{x^3-1}} = \frac{1}{3}[\log(x+2) - \log(x-1)]$.

20. $\log\sqrt{\dfrac{1}{x+1} - \dfrac{1}{x} + \dfrac{1}{x^2}} = -[\log x + \frac{1}{2}\log(x+1)]$.

21. Show that if a and b are the legs of a right triangle and c is the hypotenuse, then

$$\log a = \tfrac{1}{2}[\log(c+b) + \log(c-b)];$$
$$\log b = \tfrac{1}{2}[\log(c+a) + \log(c-a)].$$

Is there a similar formula for $\log c$?

22. In a right triangle given $c = 285$, $b = 215$, find a (see exercise 21).

23. In a right triangle given $c = 34.69$, $a = 26.21$, find b.

102. Tables of logarithms. Explanation of the method of looking up logarithms and antilogarithms in tables will not be given here. The student who is unfamiliar with the use of tables is referred to books on elementary algebra or trigonometry for a detailed discussion of this procedure. A four-place table is found on pages 212–213, together with a rule for its use.

103. Exponential equations. An equation in which the unknown occurs in an exponent is called an **exponential equation.** The use of logarithms is usually required to solve such equations. In these solutions it should be remembered that logarithms are nothing but numbers, and should be treated as such.

EXAMPLES

1. Solve $2^{x-4} = 5$.

Solution. Taking the logarithm of each member of the equation,

$$\log (2^{x-4}) = \log 5.$$

By Theorem II, p. 177, $(x - 4) \log 2 = \log 5$,

or
$$x = 4 + \frac{\log 5}{\log 2} = 4 + \frac{.6990}{.3010}$$
$$= 4 + 2.322 = 6.322.$$

It should be kept in mind that in the fraction $\dfrac{\log 5}{\log 2}$ both numerator and denominator are numbers, and that it is the quotient of these numbers which is called for. $\text{Log}\,\frac{5}{2}$ is a very different number from $\dfrac{\log 5}{\log 2}$ and should not be confused with it.

2. Solve $4^{x^2-2x} = 64$.

Solution. $4^{x^2-2x} = 4^3$.

Taking the logarithm of each member of the equation,

$$(x^2 - 2x) \log 4 = 3 \log 4.$$

Dividing by $\log 4$, $x^2 - 2x = 3$,

or $(x - 3)(x + 1) = 0.$

Hence $x = 3 \text{ or } -1.$

EXERCISES

Solve, obtaining results to four figures:

1. $2^x = 19.$
2. $(3.1)^x = 90.7.$
3. $10^{-x} = 20.$
4. $3^{1-x} = 21.45.$
5. $5^{x+2} = 7^{x-3}.$
6. $(2.2)^{3x-1} = (3.3)^{x+4}.$
7. $2^{x^2-3x} = 16.$
8. $6^{2x^2-4} = 36^x.$
9. $(4^x)^{x-2} = 2^4.$
10. $2^{x^2} = 64^x.$
11. $a^{3x-2} = b^{2x-3}.$
12. $a^{3x^2-7x+2} = 1.$
13. $\sqrt[4]{a^x} = \sqrt[3]{a^x \sqrt{a^{x-3}}}.$
14. $\sqrt[x]{a^3} = \sqrt[1-x]{a^x}.$
15. $\sqrt[2x]{243} = \sqrt[4]{3^x}.$
16. $(7.2)^{\frac{1}{x}} = (5.9)^x.$
17. $(\frac{1}{2})^{x+4} = 25^{3x+2}.$
18. $18^{3-4x} = (54\sqrt{2})^{3x-2}.$
19. $4^{2x} - 8(4^x) + 12 = 0.$
20. $16^{3x} - 16(64^x) + 64 = 0.$
21. $\sqrt[3]{a^{5x+7}} \cdot \sqrt[4]{a^{3x+10}} = a^2 \cdot \sqrt{a^{5x}}.$
22. $\sqrt[3]{\left(\frac{a}{b}\right)^x} = \sqrt[5]{\left(\frac{b}{a}\right)^{11}}.$
23. $(\frac{13}{17})^{2x-5} = (\frac{17}{13})^{5x-9}.$
24. $32^{\frac{x+5}{x-7}} = (.25)\,128^{\frac{x+17}{x-3}}.$
25. $\sqrt[7x]{7^{5x+7}} = \sqrt[5x]{5^{7x+5}}.$
26. $8^{3^x} = 6^{4^x}.$
27. $3^x - 5^{x+2} = 3^{x+4} - 5^{x+3}.$
28. $9.7^{-x} \cdot (1105.8)^x = 57^{x-1}.$
29. $5^x + 5^{x+1} + 5^{x+2} = 3^x + 3^{x+1} + 3^{x+2}.$
30. Solve the simultaneous equations:

$$3^x = y^2,$$
$$18\,y^x - y^{2x} = 81.$$

31. What difficulties are met if one attempts to solve such equations as the following: (a) $2^x + 3^x = 5^x$; (b) $x^2 = 2^x$; (c) $x^x = 2$.

104. Compound interest. Let P represent the number of dollars on interest and r the rate, expressed in hundredths. Thus if the interest is 6%, $r = .06$. Then the interest at the end of one year is $r \cdot P$ dollars, and the accumulation at the end of the year is

$$P + rP = P(1+r) \text{ dollars.}$$

The interest for the second year is $P(1+r) \cdot r$, and the entire accumulation at the end of two years is

$$P(1+r) + P(1+r) \cdot r = (1+r)(P+rP) = P(1+r)^2 \text{ dollars.}$$

Similarly, at the end of n years the accumulation is

$$A = P(1+r)^n \text{ dollars.}$$

By means of this formula A, P, r, and n are related, and if all but one of these are given, the remaining one can be found.

For example, if A, P, and r are known, n can be expressed in terms of these by first taking the logarithm of each side of the equation and solving as follows:

$$\log A = \log P + n \log(1+r), \quad \text{or} \quad n = \frac{\log A - \log P}{\log(1+r)}.$$

If the interest is compounded semiannually, at the end of the first half year it is $\left(\frac{r}{2}\right)P$, and the accumulation at that time is

$$P\left(1+\frac{r}{2}\right) \text{ dollars.}$$

Proceeding as in the case where the interest was assumed to be compounded annually, it is found that the accumulation for n years when the interest is compounded semiannually is

$$A = P\left(1+\frac{r}{2}\right)^{2n} \text{ dollars.}$$

In this case
$$n = \frac{\log A - \log P}{2\log\left(1+\frac{r}{2}\right)}.$$

EXAMPLE

Find the accumulation at the end of 10 years on \$1500 at 4%, compounded semiannually. Find the limit of error of the computation.

Solution.
$$A = P\left(1+\frac{r}{2}\right)^{2n}.$$

$$P = 1500, \quad r = .04, \quad n = 10.$$

$$A = 1500\left(1+\frac{.04}{2}\right)^{20} = 1500(1.02)^{20}.$$

$$\begin{aligned} \log 1500 &= 3.1761 \\ 20 \log 1.02 &= \underline{.1720} \\ \log A &= 3.3481 \end{aligned}$$

$$A = 2229 \text{ dollars.}$$

To determine the limit of error of this computation it is necessary to observe that the limit of error in the table of logarithms is .00005; that is, the true value of any logarithm may be greater or less than the one given in the table by not

more than this number. Hence in multiplying log 1.02 by 20, the possible error is $20 \times .00005 = .001$. In the logarithm 3.1761 there is a further possible error of .00005. Hence the total limit of error in log A is .00105; that is, the true value of log A is between 3.3470 and 3.3492. Reference to the table shows that this amount of error in the logarithm would correspond to an error of 6 in the fourth significant figure of the antilogarithm. Hence the limit of error in the result is 6 dollars. If a result correct to cents is desired, seven- or eight-place tables would be necessary.

105. Change of base. As we shall see on page 207, the computation of logarithms is actually carried out, not for the base 10 which we ordinarily use in our tables, but for the base $e = 2.7128 \cdots$. In order to pass from logarithms for one base to those for another we need the following

THEOREM. $$\log_c x = \frac{\log_b x}{\log_b c}. \tag{1}$$

Suppose that the logarithms of all real numbers have been found for the base b.

Let x be a number whose logarithm for the new base, c, is desired.

Suppose that $\log_c x = z$; that is, $c^z = x$. (2)

Taking the logarithm of each member of this equation for the base b, we have

$$z \log_b c = \log_b x,$$

or

$$z = \frac{\log_b x}{\log_b c}.$$

Hence by (2) $$\log_c x = \frac{\log_b x}{\log_b c}.$$

It will be necessary to use the foregoing theorem on page 210 in order to obtain the logarithms for the base 10 from those for the base e.

EXAMPLE

Find $\log_2 5$.

Solution. Here $x = 5, \quad b = 10, \quad c = 2.$

Applying (1), we find $\log_2 5 = \dfrac{\log 5}{\log 2} = \dfrac{.6990}{.3010} = 2.322.$

It is observed that this question is equivalent to the following: Find the power to which 2 must be raised so that the result will be 5.

EXERCISES

Find the accumulation on each of the following:

1. P dollars for n years at the rate r compounded quarterly.

2. \$1200 at the end of 8 years at 5% compounded annually. Find the limit of error of the computation.

3. \$850 at the end of 12 years at 6% compounded semiannually. Find the limit of error of the computation.

4. \$1500 at the end of 10 years at 4% compounded quarterly. Find the limit of error of the computation.

5. \$75 at the end of 6 years 8 months at 5% compounded annually.

6. In what time will a sum double itself at 4% compounded annually? at 5%?

7. At what rate will a sum double itself in 20 years, interest compounded annually?

8. At what rate will a sum treble itself in 15 years, interest compounded annually?

9. In what time will a sum double itself at 5% compounded semiannually?

10. A certain society offers a life membership for \$50, which exempts the member from further dues. Other members must pay \$5 annually. Counting interest at 5%, show that if a member lives more than 13 years after joining the society, it pays him to take out a life membership.

11. What rate of interest payable annually is equivalent to 5% payable semiannually?

12. A house worth \$5000 is let for \$400 a year, payable at the end of each quarter. If the tenant wishes to pay at the end of the year, how much must the rent be raised in order that the landlord may obtain the same rate of interest as before?

13. Find (a) $\log_2 3$; (b) $\log_3 8$; (c) $\log_{2.8}(11.98)$.

14. Find the value of the product

$$\log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9.$$

15. Seventeen is what power of 3?

16. To what power must $2\sqrt{3}$ be raised to obtain $\sqrt[3]{7}$?

CHAPTER XI

INFINITE SERIES

106. Variables. A letter which, during a given discussion, may take on several distinct values is called a *variable.* A variable need not take on all or even many numerical values. It is not uncommon to speak of x in the equation $ax^2 + bx + c = 0$ as a variable, although it can take on only two values and at the same time satisfy the equation.

It should be noted, however, that in considering the *function* $ax^2 + bx + c$, as we do, for example, when we plot it, x is a variable which takes on all real values.

In equations like $2x + 3y = 4$, both x and y take on countless values and both are called variables. Usually the values which a variable may take on are limited by some law which is frequently expressed by means of an equation.

In the equation $2x + 3y = 4$ the variation of x and y is limited to those values which satisfy the equation. For example, if x equals 8, the corresponding value of y is determined by the equation to be -4.

107. Infinity. If a variable takes on the succession of integral values 1, 2, 3, $\cdots$, we can think of no greatest value of the variable; for when we imagine a certain integer as the last one, we can immediately think of a greater. We express this condition by saying that as the variable takes on the positive integral numbers in order, it becomes infinite. To say that a variable becomes infinite is a short way of saying that a value of the variable exists which is greater than an arbitrarily chosen number M, however great M may be. Infinity is not a number, and must not be used in operations as if it were. It is merely a name to indicate that a variable has become greater than any number. It is often symbolized by ∞.

108. Limits. Consider the set of numbers

$$\tfrac{1}{2},\ \tfrac{3}{4},\ \tfrac{7}{8},\ \tfrac{15}{16},\ \cdots. \tag{1}$$

From the numbers which are written one can determine as many more as desired following the same law. As one reads toward the

right, the numbers increase, each one being the arithmetical mean of its predecessor and unity. The nth term is $\frac{2^n - 1}{2^n}$. However far we continue the set of numbers we never find one which exceeds or even equals 1. But however small a number we may think of, say .01, we can find a number in the set which differs from 1 by less than this number. The number $\frac{127}{128}$ is the first in the set which differs from 1 by less than .01. If we had thought of .001, .0001, or any other small number instead of .01, we could have found a number in the set further to the right which differed from 1 by less than it. The numbers of this set may be considered as different values which a variable x assumes. We have the following

DEFINITION. *If a variable x takes on values in order, such that the difference between x and some fixed number A becomes and remains numerically less than d, however small d may be taken, then x is said to approach A as a limit.*

In the case mentioned, the value we first took for d was .01, and we saw that $\frac{127}{128}$ differed from the fixed number 1 by less than .01. And not only this, but all numbers in the set further to the right differ from 1 by even less. As a matter of fact, this set of numbers actually approaches 1 as a limit. This can be proved by showing that for a sufficiently large value of n the value of $\frac{2^n - 1}{2^n}$ differs from 1 by as little as we please.

The reason for including the words "and remains" in the definition may be appreciated if the set of numbers (1) be replaced by a set in which the same numbers are found, except that every alternate number is replaced by the number 2, thus: $\frac{1}{2}$, 2, $\frac{7}{8}$, 2, $\frac{31}{32}$, 2, $\cdots$. Then the set would not have 1 as a limit, for although a number of the set could be found which differs from 1 by less than any assigned value, yet the very next number of the set differs from 1 by unity. These 2's may occur in the early part of the set and not affect the approach to a limit, but they cannot appear throughout the whole extent of the set.

The approach of a variable to a limit may be illustrated geometrically by considering the numbers of the set (1) as measuring distances on a line.

We have already stated that if, after taking on a certain finite number of values, the variable always remains less than the distance

d from A, however small d may be taken, it approaches A as a limit. This condition is illustrated geometrically by a bunching of the points near the point representing the limit.

If, instead of considering the variable x, we deal directly with the set of numbers, the foregoing definition may be given in another form, which lends itself more conveniently to symbolic expression as follows:

If, in the set of numbers, $u_1, u_2, u_3, \cdots, u_{n-1}, u_n, \cdots$ *a subscript* m *can be found such that the difference between any* u_n *and* A *(when* n *is greater than* m*) is numerically less than* d *(where* d *is an arbitrarily small positive number, then the set of* u*'s has* A *for a limit.*

Expressed in symbols,

If $\quad |u_n - A| < d, \quad$ for $\quad n > m, \quad$ then $\quad \lim\limits_{n=\infty} u_n = A.$

Many variables cannot take on their limiting values. Most, if not all, of the variables which one meets in elementary geometry are of this character. Such limits are sometimes called inaccessible. Other variables do take on their limiting values. For example, the distance from a falling particle to the ground is a variable which approaches and takes on the value zero. Such limits are often called accessible. But whether the limit is accessible or not, is of no consequence in the definitions given above.

We are now in a position to see that the fraction $\frac{a}{n}$ approaches zero as a limit if a is a constant and n is a variable which becomes infinite. Consider, for example, the fraction $\frac{2}{n}$. The process of determining whether this fraction approaches zero may be explained clearly by means of the following dialogue, in which Henry claims that $\frac{2}{n}$ does not approach zero, and John contends that it does.

Henry. "I see no reason why this fraction $\frac{2}{n}$ approaches zero as a limit."

John. "You must admit that it does approach zero if, when you name any number as small as you like, I can find a value of n so large that for my n and all larger values the fraction $\frac{2}{n}$ is less than your small number."

Henry. "Yes, I admit that, for it is in accordance with the definition of the limit of a variable."

John. "Well, then, name a small number."

Henry. "I challenge you to find an n which will make the fraction less than .0001."

John. "If n has the value 100,000, you will find that $\frac{2}{n}$ is less than your .0001.

Henry. "I see that; but suppose I name .000001?"

John. "Then I would let n have the value 10,000,000."

Henry. "There is no use in continuing this further, for I see that whatever small number, as k, I may name, if you take n equal to $\frac{10}{k}$, then the fraction $\frac{2}{n}$ becomes $\frac{k}{5}$, which is certainly less than the k which I named."

In a manner similar to that outlined in the foregoing dialogue it may be seen that $\frac{a}{n}$ becomes and remains less than any small number k for all values of n equal to or greater than $\frac{2a}{k}$. Hence the fraction $\frac{a}{n}$ approaches zero as a limit.

109. Infinite series. We are familiar with sums, like $a + b + c$, which have a definite number of terms. We have also used sums, like $x^n + a_1x^{n-1} + \cdots + a_n$, which have an indefinite number, n, of terms; but we have always assumed that n has a finite value, so that the operations which are indicated in any such function can actually be performed in a finite length of time. An *infinite series* is the indicated sum of a never-ending or infinite set of terms. Since we can never write down all of the terms of an infinite series, it is essential that from the few which we do write the law may be apparent by which we can find as many more as we desire.

The infinite series whose terms are $u_1, u_2, u_3, \cdots, u_n, \cdots$ is often denoted by $\sum u_n$, read "summation u_n," or by $\sum_{n=1}^{\infty} u_n$, read "summation u_n from $n = 1$ to $n = \infty$." Thus we write

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots.$$

The nth term of this series is u_n; that is, the subscript is the same as the number of the term. The sum of the first n terms is denoted by S_n.

Thus $\qquad S_n = u_1 + u_2 + u_3 + \cdots + u_n,$

and $\qquad S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3$, etc.

Sometimes an infinite series is written in the form

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots.$$

In this series the nth term is u_{n-1}, the subscript being one less than the number of the term, and u_n being the $(n + 1)$st term of this series. The sum of the first n terms is then

$$S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}.$$

Throughout this chapter n will represent any positive integer or zero.

EXAMPLES

1. Write down the first five terms of the series $\sum_{n=1}^{\infty}\frac{2n-1}{2^n}$. What is the 10th term? the nth term?

Solution. $$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}=\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\frac{9}{2^5}.$$

The 10th term is $\frac{2\cdot 10-1}{2^{10}}=\frac{19}{2^{10}}$. The nth term is $\frac{2n-1}{2^n}$.

2. Write down the first five terms of the series $\sum_{n=0}^{\infty}\frac{2n+1}{(n+1)^2}$. What is the 8th term? the nth term? the $(n+1)$st term?

Solution. $$\sum_{n=0}^{\infty}\frac{2n+1}{(n+1)^2}=1+\frac{3}{2^2}+\frac{5}{3^2}+\frac{7}{4^2}+\frac{9}{5^2}.$$

The 8th term is $$\frac{2\cdot 7+1}{8^2}=\frac{15}{8^2}.$$

The nth term is $$\frac{2(n-1)+1}{[(n-1)+1]^2}=\frac{2n-1}{n^2}.$$

The $(n+1)$st term is $$\frac{2n+1}{(n+1)^2}.$$

EXERCISES

Write down the first five terms of the following series:

1. $\sum_{n=1}^{\infty} n^2$.

2. $\sum_{n=0}^{\infty}\frac{1}{2^n}$.

3. $\sum_{n=1}^{\infty} n(n+1)$.

4. $\sum_{n=0}^{\infty}\frac{1}{n!}$.

5. $\sum_{r=1}^{\infty}(2r)^2$.

6. $\sum_{n=0}^{\infty}\frac{n+1}{(2n+1)!}$.

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$.

8. $\sum_{n=1}^{\infty}\frac{x^n}{n!}$.

9. $\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}$.

10. $\sum_{n=0}^{\infty}(-1)^n\frac{3^{n+1}}{5^{2n+1}}$.

11. $\sum_{n=2}^{\infty}\frac{2n-1}{2n}x^{n-2}$.

12. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2^k}$.

13. Write down (a) the 7th term in the series of exercise 3; (b) the 8th term in the series of exercise 6; (c) the 9th term in the series of exercise 10; (d) the 10th term in the series of exercise 12.

14. Find (a) the sum of the first five terms of the series in exercise 2; (b) the sum of the first four terms of the series in exercise 7; (c) the sum of the first six terms of the series in exercise 5.

15. How many terms of the series of exercise 2 must be taken in order to make S_n differ from 2 by less than .001?

16. How many terms of the series of exercise 5 must be taken in order to make S_n greater than 1000?

Find the nth term in each of the following series and express the series in the Σ notation:

17. $1 + 2 + 3 + 4 + \cdots$.

18. $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$.

19. $1^2 + 3^2 + 5^2 + 7^2 + \cdots$.

20. $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots$.

21. $1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$.

22. $x + x^3 + x^5 + x^7 + \cdots$.

23. $\frac{1}{x} + \frac{1}{x^4} + \frac{1}{x^9} + \frac{1}{x^{16}} + \cdots$.

24. $\frac{3}{5^2} + \frac{3^2}{10^2} + \frac{3^3}{15^2} + \frac{3^4}{20^2} + \cdots$.

25. $\frac{x^2}{\sqrt{2}} + \frac{x^4}{\sqrt{4}} + \frac{x^6}{\sqrt{6}} + \frac{x^8}{\sqrt{8}} + \cdots$.

26. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

27. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \cdots$.

28. $1 - \frac{2!}{2^2} + \frac{3!}{3^2} - \frac{4!}{4^2} + \cdots$.

29. In the series of exercise 23, compute S_8 if $x = 2$.

30. In the series of exercise 26, compute S_4 if $x = \frac{1}{2}$.

110. Convergence and divergence. An inspection of the two following series indicates that they are of quite distinct types:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots, \tag{1}$$

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots, \tag{2}$$

In (1) each term adds only one half the difference between the sum of the preceding terms and 2. Consequently, however many terms we may add together, we can never obtain a sum which exceeds or even equals 2.

Since there is an infinite number of terms in an infinite series, it would be impossible to compute their sum in less than an eternity of time. But since this is not at our disposal, it is, in strictness, without meaning to speak of the sum of the terms of an infinite series, for such an operation could never be performed. In the case of series (1), 2 is not the sum of any number of terms which we could write down; it is greater than any such sum. But it is approached as a limit by the sums of increasing numbers of terms. In spite of the fact that 2 is not really the sum of the infinite number of terms of (1), but the limit of S_n as n becomes infinite, nevertheless it is called the sum of the series.

DEFINITION. *When the sum S_n of the first n terms of an infinite series approaches a limit, as n becomes infinite, the series is said to be* **convergent**. *This limit is called the* **sum of the series.**

In most cases it is a simpler matter to find that this limit exists, and hence that the series in question is convergent, than it is to determine the exact value of the limit.

In series (2) each term is greater than the preceding one, and by adding a sufficient number of them a sum can be obtained greater than any number which we may name.

DEFINITION. *When the sum S_n of the first n terms of an infinite series does not approach a finite limit as n becomes infinite, the series is said to be* **divergent.**

Consider the series $1+2+3+\cdots+n+\cdots$.

In this case a value of n can be found so great that the value of S_n is greater than any value which can be assigned. Hence S_n does not approach a finite limit. In the case of all the divergent series considered in this text, the value of S_n becomes greater than any assigned number, provided we take n large enough.

There is another kind of divergent series of which

$$1-1+1-1+\cdots \tag{1}$$

is the type. This is called an oscillating series, because the values of S_n oscillate between certain values, but never settle down to a limiting value. In series (1) S_n is either zero or 1, according as n is even or odd.

THEOREM. *If each term of an infinite series with positive terms is greater than a fixed number, however small, the series is divergent.*

For a sufficient number of terms, each greater than this fixed small number, would add up to a sum greater than M, however great M might be.

This theorem assures us that none of the following tests for convergence are necessary unless the terms of the series approach zero as n becomes infinite.

111. Comparison test for convergence. The problem of finding whether a given series converges or not, and that of finding the exact value to which it converges, are quite distinct. We shall give some of the most important methods for attacking the former problem, but shall content ourselves with computations for obtaining the approximate value of the sum of the series.

In what follows we shall make use of the following

ASSUMPTION. *If S_n is a variable which always increases when n increases, but which never exceeds some finite number D, then S_n approaches a limit A, which cannot be greater than D.*

The only type of series for which we have hitherto derived any test for convergence is the geometrical series

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \frac{a}{1-r}, \qquad (1)$$

where r is numerically less than 1 and a is any real number (see § 10).

Consider the two series

$$S = 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n!} + \cdots,$$

$$S' = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

An inspection shows that each term of S after the second is less than the term of S' which is directly below it. Furthermore, the nth term of S, namely $\frac{1}{n!}$, is less than the nth term, $\frac{1}{2^{n-1}}$, of S', since

$$2 \cdot 3 \cdots n > 2 \cdot 2 \cdots 2, \text{ to } n-1 \text{ factors, if } n > 2.$$

But S' is a geometrical series with the limit 2. Hence we would expect S to converge to a limit not greater than 2. That this is the case follows from the following general

THEOREM. *Let $u_1 + u_2 + u_3 + \cdots$ be an infinite series of positive terms which is to be tested. If a series of positive terms $v_1 + v_2 + v_3 + \cdots$ can be found, which is known to converge, and is such that each term of the u-series is equal to or less than the corresponding term in the v-series, then the u-series must converge, and its sum is equal to or less than the sum of the v-series.*

Let the sum of the v-series be A.

Let $$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$
and $$S'_n = v_1 + v_2 + v_3 + \cdots + v_n,$$
where n is any positive integer. Then since the second series converges to A, we have
$$\lim_{n=\infty} S'_n = A.$$

Since all of the terms of the v-series are positive, we have
$$S'_n < A.$$
But, by hypothesis, $$S_n \leqq S'_n.$$
Hence $$S_n < A;$$
that is, the sum of any number of terms of the u-series is less than a fixed number. Hence by the assumption on page 192 the limit of S_n exists and is not greater than A; that is, the u-series converges.

It is often necessary to disregard some of the first terms of a series in order to apply this theorem. But since the sum of any finite number of terms must be finite, it is sufficient to show that the series after a certain number of terms converges.

To test a series of positive terms for convergence, we write down the nth term of the given series, or the nth term of the series which remains after omitting some of the first terms from the given series. Call this term u_n. Now compare this with v_n, the nth term of a series known to be convergent. If $u_n \leqq v_n$ for every value of n greater than any particular integer, the u-series is convergent. This is called the **comparison test for convergence.** The v-series is called the comparison series. If u_n does not turn out to be equal to or less than v_n, this does

not prove that the u-series is not convergent; it merely shows that the v-series used is not effective as a comparison series. Any series derived from (1), p. 192, by substituting any real number for a and any positive number less than 1 for r is known to be convergent and can be used as a comparison series. After any series has been proved convergent it can be used as a comparison series for proving other series convergent.

EXAMPLE

Test the series

$$S = 2 + 1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots.$$

Solution. Disregarding the first two terms, the nth term of the remaining series is

$$u_n = \frac{1}{(n+1)^n}.$$

Use as a comparison series the geometric series (1), p. 192, where $a = 1$, $r = \frac{1}{2}$, and the first term is dropped.

Then $$S' = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

This series is known to be convergent and its nth term is $v_n = \dfrac{1}{2^n}$. We must now show that $u_n \leqq v_n$ for all values of n greater than some integer; that is, that

$$\frac{1}{(n+1)^n} \leqq \frac{1}{2^n},$$

or $$(n+1)^n \geqq 2^n. \tag{2}$$

This is true for all values of $n > 0$, for if $n = 1$, (2) becomes $2 = 2$; if $n > 1$, evidently $(n+1)^n > 2^n$. Hence S is convergent.

EXERCISES

Test the following series:

1. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots.$
2. $\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \frac{1}{2^{16}} + \cdots.$
3. $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots.$
4. $\frac{1}{5} + \frac{1}{7} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 2^3} + \cdots.$
5. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots.$
6. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$

7. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots.$

8. $1+\frac{1}{2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\frac{1}{1+4^2}+\cdots.$

9. $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^n}.$

10. $\sum_{n=1}^{\infty}\frac{n}{(n+1)^n}.$

11. State an assumption similar to that of the preceding section regarding the limit of a variable which continually decreases, but which remains greater than a fixed number.

12. State and prove a theorem similar to that of the preceding section regarding the convergence of a series, each of whose terms is negative.

112. The Harmonic Series. One of the most important series for the purposes of testing divergence is the Harmonic Series,

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\cdots.$$

The terms of this series become smaller and smaller and approach zero as a limit. It is difficult to believe at first sight that the sum of terms of this character can add up to a value greater than any number which we can assign. But we can prove the

THEOREM. *The Harmonic Series is divergent.*

Consider the terms of this series grouped as follows, the successive parentheses containing 1, 2, 4, 8, 16, $\cdots$ terms respectively:

$$(\tfrac{1}{2})+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+\cdots.$$

Since in the second parenthesis $\frac{1}{3}$ is greater than $\frac{1}{4}$, their sum is greater than twice $\frac{1}{4}$, or $\frac{1}{2}$. Similarly, in the third parenthesis, the sum is greater than four times $\frac{1}{8}$, or $\frac{1}{2}$; that is, by arranging the series in this way we see that it consists of the sum of an infinite number of groups of terms, each of which is greater than $\frac{1}{2}$. Hence the sum of the series does not exist, and the series is divergent.

113. Comparison test for divergence. We can now compare a series with the Harmonic Series and obtain a test for divergence similar to that for convergence in § 111. For example, consider the two series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

and

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}+\cdots.$$

Since the denominator of each fraction in the second series is less than the denominator in the term directly above it, each term of the second series is greater than the corresponding term of the first. But the first series diverges; hence one would expect the second to diverge also. That this is the case appears from the following

THEOREM. *Let $u_1 + u_2 + u_3 + \cdots$ be an infinite series of positive terms which is to be tested. If a series of positive terms, $v_1 + v_2 + v_3 + \cdots$, can be found which is known to diverge, and is such that each term of the u-series is equal to or greater than the corresponding term of the v-series, then the u-series must also diverge.*

Suppose we have $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$

and $$v_1 + v_2 + v_3 + \cdots + v_n + \cdots,$$

and suppose that the v-series diverges; that is, that a value of n can be found such that
$$v_1 + v_2 + v_3 + \cdots + v_n > M,$$

where M is a number taken arbitrarily large.

By hypothesis $u_k \geqq v_k$ for every integral value of k. From these hypotheses it follows immediately that

$$u_1 + u_2 + u_3 + \cdots + u_n \geqq M;$$

that is, that the u-series diverges.

The preceding theorem shows that in order to prove that a series is divergent we may show that its nth term, u_n, is equal to or greater than the nth term, v_n, of a known divergent series, for all values of n greater than some integer. This is called the **comparison test for divergence.**

Besides the Harmonic Series a useful one to employ in testing for divergence is the geometrical series in the case where the ratio is greater than unity. This may be written in the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots, \qquad r > 1.$$

Here a may be given any convenient numerical value and r any value > 1. The series is then known to be divergent.

As in the test for convergence it is often necessary to neglect the first few terms in the application of the theorem of this section. In so doing we merely recognize that if a series from a certain term on diverges, the entire series must diverge.

EXAMPLE

Test the series $1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots.$

Solution. The nth term of this series is $u_n=\dfrac{n-1}{n}$.

Comparing this with the nth term of the Harmonic Series, $v_n=\dfrac{1}{n}$, we will show that

$$\frac{n-1}{n}\geqq\frac{1}{n}, \tag{1}$$

when n is greater than some particular integer. This is true when $n>1$; for when $n=2$, (1) becomes $\frac{1}{2}=\frac{1}{2}$, and when $n>2$, evidently $n-1>1$. Hence the series is divergent.

EXERCISES

Test the following series:

1. $\dfrac{3}{2}+\dfrac{5}{4}+\dfrac{9}{8}+\dfrac{17}{16}+\cdots.$

2. $1+\dfrac{1}{2^r}+\dfrac{1}{3^r}+\dfrac{1}{4^r}+\cdots,\quad r<1.$

3. $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{6}+\dfrac{1}{8}+\cdots.$

4. $1+\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\cdots.$

5. $1+\dfrac{3}{1\cdot 2}+\dfrac{5}{2\cdot 3}+\dfrac{7}{3\cdot 4}+\dfrac{9}{4\cdot 5}+\cdots.$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n}{1+n^2}.$

114. Indeterminate forms. In the sections which follow it will be necessary to consider the limit approached by a fraction when both numerator and denominator become infinite.

Consider, for example, the fraction $\dfrac{n+2}{2n+1}$. If n takes on the positive integral values in order, we can find a value of n so large that both numerator and denominator of the fraction are greater than any definite number, like 1000 or 1,000,000, which we can imagine; that is, both numerator and denominator of the fraction become infinite as n becomes infinite. The limit of the fraction, when written in this form, does not seem to have any definite numerical value. But let us divide both numerator and denominator of the fraction by n before we begin to let n increase in value. We then obtain

$\dfrac{n+2}{2n+1}=\dfrac{1+\dfrac{2}{n}}{2+\dfrac{1}{n}}$. If in the fraction in the right member of this equation we let n take on larger and larger values, it appears that

the fractions $\frac{1}{n}$ and $\frac{2}{n}$ approach zero as n becomes infinite, leaving $\frac{1}{2}$ as the value which the original fraction approaches. This fact we write as follows: $\lim\limits_{n=\infty} \frac{n+2}{2n+1} = \frac{1}{2}$.

Sometimes we can accomplish a similar result by other means. Suppose we have given the fraction $\frac{2n+4}{n+2}$. If n becomes infinite, both numerator and denominator of the fraction also become infinite. But if one observes that $\frac{2n+4}{n+2} = \frac{2(n+2)}{n+2}$, and the factor $n+2$ is canceled from numerator and denominator before n begins to take on the increasing values, it appears that the value of the fraction is always 2, and that this value is entirely independent of what value n may have. Hence $\lim\limits_{n=\infty} \frac{2n+4}{n+2} = 2$.

The point in each of these methods consists in throwing the variable into a position so that the limit of the fraction can be found when the variable finally becomes infinite.

EXAMPLES

1. Find $\lim\limits_{n=\infty} \frac{n^2 - 7n + 2}{3n^2 + 1}$.

Solution. Dividing numerator and denominator of the fraction by n^2, we obtain

$$\frac{n^2 - 7n + 2}{3n^2 + 1} = \frac{1 - \frac{7}{n} + \frac{2}{n^2}}{3 + \frac{1}{n^2}}.$$

Letting n become infinite, each of the fractions $\frac{7}{n}$, $\frac{2}{n^2}$, and $\frac{1}{n^2}$ approaches 0, and the original fraction approaches $\frac{1}{3}$.

Hence

$$\lim_{n=\infty} \frac{n^2 - 7n + 2}{3n^2 + 1} = \frac{1}{3}.$$

2. Find $\lim\limits_{n=\infty} \frac{n^2 + 2n + 1}{n^3 - 3}$.

Solution. Dividing by n^3, we obtain

$$\frac{n^2 + 2n + 1}{n^3 - 3} = \frac{\frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}}{1 - \frac{3}{n^3}}.$$

Letting n become infinite, the numerator approaches zero, while the denominator approaches 1.

Hence $$\lim_{n=\infty}\frac{n^2+2n+1}{n^3-3}=\frac{0}{1}=0.$$

3. Find $\lim\limits_{n=\infty}\frac{(n+1)!}{n!}$.

Solution. $$\frac{(n+1)!}{n!}=n+1.$$

Hence $$\lim_{n=\infty}\frac{(n+1)!}{n!}=\infty.$$

EXERCISES

Evaluate the following limits:

1. $\lim\limits_{n=\infty}\frac{n+1}{2n}$.

2. $\lim\limits_{n=\infty}\frac{n}{n^2+1}$.

3. $\lim\limits_{n=\infty}\frac{\sqrt{2n+1}}{n\sqrt{n}}$.

4. $\lim\limits_{n=\infty}\frac{n!}{(n+1)!}$.

5. $\lim\limits_{n=\infty}\frac{n^2-1}{(n+1)^2}$.

6. $\lim\limits_{n=\infty}\frac{3n^2-6}{n^3-2n}$.

7. $\lim\limits_{n=\infty}\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}}$.

8. $\lim\limits_{n=\infty}\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}}$.

9. $\lim\limits_{n=\infty}\frac{(n+1)^2}{n(n+2)}$.

10. $\lim\limits_{n=\infty}\frac{n^2(n+1)!}{(n+1)^2 n!}$.

11. $\lim\limits_{n=\infty}\frac{5(4n)^3}{4(n+1)^3}$.

12. $\lim\limits_{n=\infty}\frac{a^2(n^6-a^6)}{n^2(n^4+n^2a^2+a^4)}$.

13. $\lim\limits_{n=\infty}\frac{n(1\cdot 3\cdot 5\cdots 2n-1)}{1\cdot 3\cdot 5\cdots 2n+1}$.

14. $\lim\limits_{n=\infty}\frac{2\cdot 4\cdot 6\cdots 2n+2}{(n+1)(2\cdot 4\cdot 6\cdots 2n)}$.

115. Ratio test. The test which in many cases is the most powerful and at the same time is the simplest to apply is described in the following

THEOREM. *Let $u_1+u_2+u_3+\cdots+u_n+\cdots$ be an infinite series of positive terms which is to be tested.*

I. *If* $\lim\limits_{n=\infty}\frac{u_{n+1}}{u_n}<1$, *the series is convergent.*

II. *If* $\lim\limits_{n=\infty}\frac{u_{n+1}}{u_n}>1$, *the series is divergent.*

III. *If* $\lim\limits_{n=\infty}\frac{u_{n+1}}{u_n}=1$, *the test fails to give us any information.*

I. Suppose that $\lim\limits_{n=\infty} \dfrac{u_{n+1}}{u_n} = t$, where t is a constant the precise value of which we need not specify further than to say that it is positive and is less than 1. From the definition of the limit of a variable (p. 186) we know that for all values of n equal to or greater than some integer m, the variable ratio $\dfrac{u_{n+1}}{u_n}$ differs from its limit by as little as we please. Let r be a number greater than t but less than 1. Then we can find an m so large that for all n's equal to or

greater than m the variable ratio $\dfrac{u_{n+1}}{u_n}$ will differ from its limit t by less than the quantity $r - t$; that is, each ratio, for values of the subscript greater than m, will be less than r. Symbolically expressed, $\dfrac{u_{n+1}}{u_n} < r$, when $n \geqq m$. Letting n take on the values $m, m+1, m+2, \cdots$, we have, then,

$$\frac{u_{m+1}}{u_m} < r, \quad \text{or} \quad u_{m+1} < ru_m,$$

$$\frac{u_{m+2}}{u_{m+1}} < r, \quad \text{or} \quad u_{m+2} < ru_{m+1} < r^2u_m,$$

$$\frac{u_{m+3}}{u_{m+2}} < r, \quad \text{or} \quad u_{m+3} < ru_{m+2} < r^3u_m.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Adding the inequalities on the right, we have

$$\begin{aligned} u_{m+1} + u_{m+2} + u_{m+3} + \cdots &< ru_m + r^2u_m + r^3u_m + \cdots \\ &< u_m(r + r^2 + r^3 + \cdots). \end{aligned}$$

The expression inside the parenthesis is a geometrical series, and since $r < 1$, it converges to some limit, say, L. Then we have

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots < u_m \cdot L.$$

Hence the original series converges (p. 192).

II. If t is greater than 1, we may take r less than t but greater than 1, and by the definition of the limit of a variable we can find a value, m, of n so great that for it and all greater values the ratio

in question will differ from t by less than the quantity $t-r$ and hence be greater than r; that is, $\frac{u_{m+1}}{u_m}>r$, where $r>1$.

It follows that $u_{m+1}>ru_m$; that is, each term is greater than the preceding one. Hence the series must diverge (§ 110).

III. When $t=1$ we are unable to determine by this method whether the series converges or diverges.

The present test fails, for example, for the Harmonic Series, though we have proved that the series diverges. It also fails for the series in exercise 5, p. 194, although the series is convergent.

EXAMPLE

Test the series $$1+\frac{2^2}{2\,!}+\frac{3^2}{3\,!}+\frac{4^2}{4\,!}+\cdots.$$

Solution. $$u_n=\frac{n^2}{n\,!},\quad u_{n+1}=\frac{(n+1)^2}{(n+1)\,!},$$

where u_{n+1} is obtained from u_n by replacing n by $n+1$.

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^2}{(n+1)\,!}\cdot\frac{n\,!}{n^2}=\frac{n+1}{n^2},$$

since $(n+1)\,!=(n+1)\cdot n\,!$,

$$\lim_{n=\infty}\frac{u_{n+1}}{u_n}=\lim_{n=\infty}\frac{n+1}{n^2}=\lim_{n=\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)=0<1.$$

Hence the series is convergent.

EXERCISES

Determine whether the following series are convergent or divergent:

1. $1+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\cdots.$

2. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots.$

3. $1+\frac{2\,!}{2^2}+\frac{3\,!}{3^2}+\frac{4\,!}{4^2}+\cdots.$

4. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots.$

5. $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\cdots.$

6. $1+\frac{2^2}{2\,!}+\frac{3^3}{3\,!}+\frac{4^4}{4\,!}+\cdots.$

7. $\frac{2^{10}}{2}+\frac{3^{10}}{2^2}+\frac{4^{10}}{2^3}+\frac{5^{10}}{2^4}+\cdots.$

8. $\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{3}{4\cdot 5}+\cdots.$

9. $\dfrac{3}{4^3}+\dfrac{3^2}{8^3}+\dfrac{3^3}{12^3}+\dfrac{3^4}{16^3}+\cdots$.

10. $1+\dfrac{1}{2^3}+\dfrac{2^2}{3^4}+\dfrac{3^3}{4^5}+\cdots$.

11. $\sum\limits_{n=1}^{\infty}\dfrac{3^n}{2^{4n}}$.

12. $\sum\limits_{n=1}^{\infty}\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{3\cdot 6\cdot 9\cdots 3n}$.

13. $1+\dfrac{1}{10}+\dfrac{1\cdot 4}{10\cdot 20}+\dfrac{1\cdot 4\cdot 7}{10\cdot 20\cdot 30}+\cdots$.

14. $1+\dfrac{1}{2^2}+\dfrac{1\cdot 3}{2!\cdot 2^4}+\dfrac{1\cdot 3\cdot 5}{3!\cdot 2^6}+\cdots$.

116. Alternating series. Up to the present all of our tests for convergence hold only for series with positive terms. The simplest case where some of the terms of a series are negative is that of an alternating series; that is, a series in which the signs of the terms alternate. For this case we have the

THEOREM. *If the absolute value of each term of an alternating series is less than that of the preceding term, and if the limit of the nth term is zero as n becomes infinite, the series converges.*

Given the series

$$S=a_1-a_2+a_3-a_4+a_5-a_6+\cdots,$$

where the a's are positive, $a_{n+1}<a_n$, and $\lim\limits_{n=\infty} a_n=0$.

Consider the two following methods of adding up the terms of the series to obtain the sum of the first n and $n+1$ terms respectively.

Assuming first that n is an even integer, we may write

$$S_n=(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+\cdots+(a_{n-1}-a_n), \qquad (1)$$

and

$$S_{n+1}=a_1-(a_2-a_3)-(a_4-a_5)-(a_6-a_7)-\cdots-(a_n-a_{n+1}). \quad (2)$$

Since the a's decrease in value as n increases, each of the parentheses in (2) contains a positive number. But since they are all subtracted from a_1, the sum of $n+1$ terms of S cannot exceed a_1, however great n may be.

An inspection of (1) shows that S_n is the sum of positive terms, and since it differs from S_{n+1} by a_{n+1}, which can be made arbitrarily small, it follows that S_n is less than a positive constant, and hence approaches a limit by the assumption on page 192.

Now the difference between S_n and its limit can be made less than any assigned value for a sufficiently large even value of n. But since

$$S_{n+1} = S_n + a_{n+1},$$

it appears that
$$\lim_{n=\infty} S_{n+1} = \lim_{n=\infty} S_n + \lim_{n=\infty} a_{n+1}.$$

Hence since $\lim\limits_{n=\infty} a_{n+1} = 0$, the series S converges when n becomes infinite by taking on all integral values in succession.

In computing the approximate value of the sum of an alternating series by adding together the first few terms, the foregoing method of proof assures us that the error in stopping the computation with any term, as a_k, does not exceed the value of the next term, a_{k+1}. For the part which is disregarded is an alternating series, and an inspection of (2) shows that the sum of a convergent alternating series cannot exceed its first term.

117. Series with positive and negative terms. When the signs in a series are not alternately plus and minus, we may often settle the question of convergence by

THEOREM I. *A series,* $u_1 + u_2 + u_3 + \cdots$, *some of whose terms are negative, is convergent if the series formed by the absolute values of the terms is convergent.*

If the series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is given, where some of the terms are positive and others are negative, the series formed by the absolute values of these terms may be denoted by $|u_1| + |u_2| + |u_3| + \cdots + |u_n| + \cdots$. All of these terms are positive quantities.

Thus the series formed by the absolute values of the terms of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots \text{ is } 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots.$$

Proof. Let the series of absolute values be $a_1 + a_2 + a_3 + \cdots$, and let it be convergent to the number A. It is certain that if in this series some of the signs are changed to minus, the resulting series will converge to some value not greater than A; for the sum of the absolute values of such terms must have a definite value. Even if we should change the signs of all the a's, the resulting series would converge to the value $-A$. But the u-series may be obtained by changing the signs of properly selected terms of the series of absolute values; hence the u-series converges.

We can now extend the proof of the ratio test for convergence so that it will apply to the case where some of the terms of the series are negative.

THEOREM II. *An infinite series $u_1 + u_2 + u_3 + \cdots$, consisting of positive and negative terms, converges if*

$$\lim_{n=\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1. \tag{1}$$

Since $\left|\frac{u_{n+1}}{u_n}\right| = \frac{|u_{n+1}|}{|u_n|}$, it follows that if $\left|\frac{u_{n+1}}{u_n}\right| < 1$, then $\frac{|u_{n+1}|}{|u_n|} < 1$. This latter inequality is the condition that the series $|u_1| + |u_2| + |u_3| + \cdots$ converges; and if this series converges, then, by Theorem I, the original series converges.

Similarly, we may prove that if $\left|\frac{u_{n+1}}{u_n}\right| > 1$, the series $u_1 + u_2 + u_3 + \cdots$ diverges.

118. Power series. The infinite series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$$

where the a's do not contain x, is called a power series in x. When the a's are numerically given, such a series may converge for certain values of x and diverge for others. For example, when the a's are each equal to 1, we have the geometrical series, which converges when $|x| < 1$ and diverges for other values of x. The ratio test may be used to determine for what values of x a given series converges.

EXAMPLE

For what values of x is the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ convergent? divergent?

Solution. $$u_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}; \quad u_{n+1} = (-1)^n\frac{x^{2(n+1)-1}}{2(n+1)-1}.$$

$$\frac{u_{n+1}}{u_n} = -\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}} = -\frac{2n-1}{2n+1}x^2.$$

$$\lim_{n=\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n=\infty}\frac{2n-1}{2n+1}x^2 = \lim_{n=\infty}\frac{2-\frac{1}{n}}{2+\frac{1}{n}}x^2 = x^2.$$

Hence the series will be convergent when $x^2 < 1$, that is, when $-1 < x < 1$; and divergent when $x^2 > 1$, that is, when $x < -1$ or > 1.

The ratio test gives no information concerning the convergence of the series when $x = 1$ or -1. A separate investigation is necessary to determine what happens for these values of x.

When $x = 1$ the series becomes

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots.$$

This is an alternating series and

$$\lim_{n=\infty} |u_n| = \lim_{n=\infty} \frac{1}{2n-1} = 0.$$

Therefore the series converges by § 116.

When $x = -1$ the series becomes

$$-(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots).$$

This series is also convergent, as it is the negative of the preceding series. Hence the original series is convergent when $-1 \leqq x \leqq 1$ and divergent for all other values of x.

EXERCISES

For what values of x are the following series convergent? divergent?

1. $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots.$

2. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

3. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

4. $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \cdots.$

5. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$

6. $1 - x + x^2 - x^3 + \cdots.$

7. $1 + x + 2!\,x^2 + 3!\,x^3! + \cdots.$

8. $1 - \frac{x}{3} + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \cdots.$

9. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

10. $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots.$

11. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots.$

12. $x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3\, x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5\, x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$

13. $\sum_{n=1}^{\infty} (-1)^{n+1}(3x)^n.$

14. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}.$

15. $\sum_{n=1}^{\infty} \frac{1}{1 + x^n}.$

16. $\sum_{n=1}^{\infty} \frac{n^p x^n}{n!}$, where p is any finite number.

119. Important special series. In the calculus a general method will be derived for expressing any ordinary function in terms of a series. The rational integral functions are really series with a finite number of terms, but rational nonintegral functions and the functions of trigonometry give rise to infinite series. The most important of these series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1}$$

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \tag{2}$$

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots. \tag{3}$$

$$\log_e \left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right]. \tag{4}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{5}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{6}$$

$$\sin^{-1} x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \cdots. \tag{7}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots. \tag{8}$$

In the trigonometric functions the unit of angle is the radian in each case. Each of these series has been shown to be convergent for certain values of x in the preceding exercises.

120. Approximate computation of the sum of a series. It is, in general, impossible to find the value of the sum of a power series for a particular value of x in terms of rational numbers, or even radicals. Fortunately it is unnecessary, for we can usually compute the value of the sum to as many places of decimals as we please, and determine the limit of error of the computation. We may find the limit of error by determining a convergent series whose sum is greater than the sum of the series which we neglect by breaking off our computation with any particular term.

This is illustrated in the computation of natural logarithms, that is, of logarithms for the base e, by the use of series (4) of the preceding section; namely,

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right].$$

This series is used for the computation of logarithms instead of (2) or (3) because it contains only terms with odd exponents, and hence converges much more rapidly than they do.

In this series we make the substitution

$$\frac{1+x}{1-x} = \frac{n+1}{n};$$

where n is a positive integer.

This gives

$$n + nx = n - nx + 1 - x,$$
$$2\,nx + x = 1,$$
$$x = \frac{1}{2\,n+1}.$$

Hence $\log_e\left(\frac{1+x}{1-x}\right) = \log_e\left(\frac{n+1}{n}\right) = \log_e(n+1) - \log_e n,$

and the series becomes

$$\log_e(n+1)$$
$$= \log_e n + 2\left[\frac{1}{2\,n+1} + \frac{1}{3(2\,n+1)^3} + \frac{1}{5(2\,n+1)^5} + \cdots\right]. \quad \text{(L)}$$

By means of this series we may compute the natural logarithm of any positive integer, $n+1$, if we know the logarithm of n for the base e.

In order to find the logarithm of a number for the base 10 from the logarithm for the base e, we apply the theorem on the change of base, § 105; namely, $\log x = \dfrac{\log_e x}{\log_e 10}$.

We shall find in exercise 14, p. 210, that $\log_e 10 = 2.303$. Hence to find the logarithm of any number for the base 10 we divide its logarithm for the base e by 2.303. Since it is simpler to multiply than it is to divide, it is customary to multiply the natural logarithm by $\dfrac{1}{\log_e 10} = .4343$, which is called the **modulus** of the system of common logarithms.

EXAMPLES

1. Find the sum of the series

$$x+\frac{x^2}{1\cdot 2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{3\cdot 4}+\cdots+\frac{x^n}{(n-1)n}+\cdots,$$

for the value $x=\frac{1}{5}$, correct to four decimal places.

Solution. Replacing x in each of the first four terms by $\frac{1}{5}$, we obtain

$$\frac{1}{5}=.2,$$

$$\frac{1}{25\cdot 2}=.02,$$

$$\frac{1}{125\cdot 6}=.001333\cdots,$$

$$\frac{1}{625\cdot 12}=\frac{.000133\cdots}{.221466\cdots}.$$

This gives us .2215 as the sum of the first four terms.

We will now determine an upper limit to the sum of the infinite series

$$\frac{x^5}{4\cdot 5}+\frac{x^6}{5\cdot 6}+\cdots,$$

which the foregoing computation neglects, in order to find out whether the series neglected would affect the fourth place in the sum .2215. We use the principle that decreasing the denominator of a fraction increases the fraction.

Hence
$$\frac{x^5}{4\cdot 5}+\frac{x^6}{5\cdot 6}+\frac{x^7}{6\cdot 7}+\cdots=x^5\left(\frac{1}{4\cdot 5}+\frac{x}{5\cdot 6}+\frac{x^2}{6\cdot 7}+\cdots\right)$$

$$<\frac{x^5}{4\cdot 5}(1+x+x^2+\cdots)$$

by § 10,
$$<\frac{x^5}{4\cdot 5}\left(\frac{1}{1-x}\right)\ (\text{if } |x|<1)$$

substituting $\frac{1}{5}$ for x,
$$<\frac{1}{3125.20}\cdot\frac{5}{4}=.00002.$$

Hence the series neglected does not affect the fourth decimal place, and the result .2215 is correct to the required four places.

2. In computing $\log_e(n+1)$ by means of series (L) find the limit of error if only the first r terms of the series in the brackets are used.

Solution. The series in brackets is

$$\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots+\frac{1}{(2r-1)(2n+1)^{2r-1}}+\cdots,$$

where the last term written is the rth term.

The remainder of the series after the rth term is

$$R_r = \frac{1}{(2r+1)(2n+1)^{2r+1}} + \frac{1}{(2r+3)(2n+1)^{2r+3}} + \frac{1}{(2r+5)(2n+1)^{2r+5}} + \cdots$$
$$< \frac{1}{(2r+1)(2n+1)^{2r+1}}\left\{1 + \frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \cdots\right\}.$$

The geometrical series in the braces has a ratio $\frac{1}{(2n+1)^2}$ which is less than 1, since n is a positive integer. Hence its sum is (§ 10)

$$S_\infty = \frac{1}{1 - \frac{1}{(2n+1)^2}}.$$

Now, since $n \geqq 1$, we have

$$2n + 1 \geqq 3,$$
$$\frac{1}{(2n+1)^2} \leqq \frac{1}{9},$$
$$1 - \frac{1}{(2n+1)^2} \geqq \frac{8}{9},$$
$$S_\infty = \frac{1}{1 - \frac{1}{(2n+1)^2}} \leqq \frac{9}{8}.$$

Hence
$$R_r < \frac{9}{8(2r+1)(2n+1)^{2r+1}}.$$

Now multiplying by 2, the coefficient of the bracket in (L), we find for the limit of error,

$$E = \frac{9}{4(2r+1)(2n+1)^{2r+1}}.$$

In computing the value of a logarithm by means of series (L), the error in using only the first r terms of the series in brackets is always less than E.

3. Compute the value of $\log_e 2$ correct to three decimal places and show that the remainder of the series after three terms does not affect the third decimal place.

Solution. Substituting $n = 1$ in series (L), we have, since $\log 1 = 0$,

$$\log_e 2 = 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \cdots\right].$$

Now
$$\frac{1}{3} = .3333$$
$$\frac{1}{3^4} = .0123$$
$$\frac{1}{5 \cdot 3^5} = .0008$$
$$.3464$$
$$2$$
$$\log_e 2 = .693$$

Substituting $n = 1$, $r = 3$, in the expression for E, we find

$$E = \frac{9}{4 \cdot 7 \cdot 3^7} = \frac{9}{61236} = .00014.$$

Thus we see that the error in neglecting all terms of the series after the third is not greater than .0002. Hence this remainder does not affect the third decimal place of the result, and $\log_e 2 = .693$ correct to three decimal places.

EXERCISES

In the following exercises make use of the series in § 119. In each case show that the terms of the series which are neglected in the computation do not affect the result:

1. Find the value of e correct to three decimal places.

2. Find the value of $e^{.05}$ correct to four decimal places.

3. Find the value of $\log_e 2$ correct to five decimal places.

4. Find the value of $\log_e 3$ correct to four decimal places.

5. Find the value of $\cos 1°$ correct to four decimal places.

HINT. $1° = \frac{\pi}{180}$ radian $= \frac{3.1416}{180} = .0175$ correct to four places, or $\frac{7}{400}$ radians. Let $x = \frac{7}{400}$ in series (6).

6. Find the value of $\sin 1°$ correct to four decimal places.

7. Find the value of $\sin 5°$ correct to four decimal places.

8. Find the value of $\cos 5°$ correct to four decimal places.

9. Using the results of exercises 7 and 8, find by division the value of $\tan 5°$ to three decimal places.

10. Find the value of π correct to two decimal places by letting $x = \frac{1}{2}$ in series (7).

11. Compute correct to five decimal places the value of $\tan^{-1}(.1)$.

12. The series of exercise 14, p. 202, gives the value of $\sqrt{2}$. Check this to three decimal places.

13. Compute the logarithms for the base e of the positive integers up to and including ten, correct to three decimal places.

14. Compute the logarithms of the first nine positive integers for the base 10, correct to three decimal places.

TABLES

Table of Logarithms

Rule for determining the characteristic of the logarithm of a number.

I. *The characteristic of a number greater than* 1 *is one less than the number of digits to the left of the decimal point.*

II. *The characteristic of a number less than* 1 *is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.*

Rule for determining the mantissa of the logarithm of a number.

Prefix the proper characteristic to the mantissa of the first three significant figures of the given number.

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference, column D of the table) by the remaining figures of the number preceded by a decimal point.

Add the product to the logarithm of the first three figures; taking the nearest decimal in the fourth place.

Rule for finding the antilogarithm.

Write the number of three figures corresponding to the lesser of the two mantissas between which the given mantissa lies.

Subtract the lesser mantissa from the given mantissa and divide the remainder by the tabular difference to one decimal place.

Annex this figure to the three already found and place the decimal point where indicated by the characteristic.

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

Square Root

To find the square root of a number with an even number of digits to the left of the decimal point, use Table I.

To find the square root of a number with an odd number of digits to the left of the decimal point, use Table II. If the number contains three significant figures, interpolate in order to make the correction for the third place as in the use of the logarithmic tables.

If the decimal point of a number is so placed that by shifting it to the right or to the left over two (or over any multiple of two) digits it comes in one of the places mentioned above, use the table which corresponds to that case.

Thus $\sqrt{24} = 4.899$; $\sqrt{2400} = 48.99$; $\sqrt{.24} = .4899$; $\sqrt{24.3} = 4.929$. Each of the foregoing is from Table I. The last requires interpolation.

Each of the following is from Table II: $\sqrt{2.4} = 1.549$; $\sqrt{240} = 15.49$; $\sqrt{.024} = .1549$; $\sqrt{24300} = 155.9$.

TABLE I

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0	0.000	1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.899	9.950

TABLE II

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	.316	.447	.548	.632	.707	.775	.837	.894	.949
1	1.000	1.049	1.095	1.140	1.183	1.225	1.265	1.304	1.342	1.378
2	1.414	1.449	1.483	1.517	1.549	1.581	1.612	1.643	1.673	1.703
3	1.732	1.761	1.789	1.817	1.844	1.871	1.897	1.924	1.949	1.975
4	2.000	2.025	2.049	2.074	2.098	2.121	2.145	2.168	2.191	2.214
5	2.236	2.258	2.280	2.302	2.324	2.345	2.366	2.387	2.408	2.429
6	2.449	2.470	2.490	2.510	2.530	2.550	2.569	2.588	2.608	2.627
7	2.646	2.665	2.683	2.702	2.720	2.739	2.757	2.775	2.793	2.811
8	2.828	2.846	2.864	2.881	2.898	2.915	2.933	2.950	2.966	2.983
9	3.000	3.017	3.033	3.050	3.066	3.082	3.098	3.114	3.130	3.146

Cube Root

If the cube root of a number with two digits to the left of the decimal point is wanted, use Table I. If the cube root of a number with one digit to the left of the decimal point is wanted, use Table II. If the cube root of a number with three digits to the left of the decimal point is wanted, use Table III.

If the decimal point of a number is so placed that by shifting it to the right or to the left over three (or any multiple of three) digits it comes into one of the places mentioned above, use the table which corresponds to that case.

Thus $\sqrt[3]{22} = 2.802$; $\sqrt[3]{22000} = 28.02$; $\sqrt[3]{.000022} = .02802$; $\sqrt[3]{22.6} = 2.827$. Each of the foregoing is from Table I. The last requires interpolation.

From Table II we obtain $\sqrt[3]{2.2} = 1.301$; $\sqrt[3]{2200} = 13.01$; $\sqrt[3]{.00226} = .1312$. The last requires interpolation.

From Table III we obtain $\sqrt[3]{.22} = .604$; $\sqrt[3]{220} = 6.04$; $\sqrt[3]{226000} = 60.9$.

TABLE I

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0	0.000	1.000	1.260	1.442	1.587	1.710	1.817	1.913	2.000	2.080
1	2.154	2.224	2.289	2.351	2.410	2.466	2.520	2.571	2.621	2.668
2	2.714	2.759	2.802	2.844	2.884	2.924	2.962	3.000	3.037	3.072
3	3.107	3.141	3.175	3.208	3.240	3.271	3.302	3.332	3.362	3.391
4	3.420	3.448	3.476	3.503	3.530	3.557	3.583	3.609	3.634	3.659
5	3.684	3.708	3.733	3.756	3.780	3.803	3.826	3.849	3.871	3.893
6	3.915	3.936	3.958	3.979	4.000	4.021	4.041	4.062	4.082	4.102
7	4.121	4.141	4.160	4.179	4.198	4.217	4.236	4.254	4.273	4.291
8	4.309	4.327	4.344	4.362	4.380	4.397	4.414	4.431	4.448	4.465
9	4.481	4.498	4.514	4.531	4.547	4.563	4.579	4.595	4.610	4.626

TABLE II

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	.464	.585	.669	.737	.794	.843	.888	.928	.965
1	1.000	1.032	1.063	1.091	1.119	1.145	1.170	1.193	1.216	1.239
2	1.260	1.281	1.301	1.320	1.339	1.357	1.375	1.392	1.409	1.426
3	1.442	1.458	1.474	1.489	1.504	1.518	1.533	1.547	1.560	1.574
4	1.587	1.601	1.613	1.626	1.639	1.651	1.663	1.675	1.687	1.698
5	1.710	1.721	1.732	1.744	1.754	1.765	1.776	1.786	1.797	1.807
6	1.817	1.827	1.837	1.847	1.857	1.866	1.876	1.885	1.895	1.904
7	1.913	1.922	1.931	1.940	1.949	1.957	1.966	1.975	1.983	1.992
8	2.000	2.008	2.017	2.025	2.033	2.041	2.049	2.057	2.065	2.072
9	2.080	2.088	2.095	2.103	2.110	2.118	2.125	2.133	2.140	2.147

TABLE III

	0	1	2	3	4	5	6	7	8	9
.0	0.000	.215	.271	.311	.342	.368	.391	.412	.431	.448
.1	.464	.479	.493	.507	.519	.531	.543	.554	.565	.575
.2	.585	.594	.604	.613	.621	.630	.638	.646	.654	.662
.3	.669	.677	.684	.691	.698	.705	.711	.718	.724	.731
.4	.737	.743	.749	.755	.761	.766	.772	.777	.783	.788
.5	.794	.799	.804	.809	.814	.819	.824	.829	.834	.839
.6	.843	.848	.853	.857	.862	.866	.871	.875	.879	.884
.7	.888	.892	.896	.900	.905	.909	.913	.917	.921	.924
.8	.928	.932	.936	.940	.944	.947	.951	.955	.958	.962
.9	.965	.969	.973	.976	.980	.983	.986	.990	.993	.997

INDEX

(The numbers refer to pages.)

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Math 24 2/5/20	
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